Globular Multicategories with Homomorphism Types

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To my parents, with thanks for all their love and support throughout my studies. Deo gratias.

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Abstract

We introduce various notions of globular multicategory with homomorphism types. We develop a higher dimensional modules construction that constructs globular multicategories with strict homomorphism types. We illustrate how this construction is related to iterated enrichment. We show how various collections of "higher category-like" objects give rise to globular multicategories with homomorphism types. We show how these structures suggest a new globular approach to the semantics of (directed) homotopy type theory.

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Chapter 1

Introduction

There are a great variety of approaches to defining a theory of weak higher categories (see for instance [32]). A *topological* approach using tools of abstract homotopy theory has enjoyed great popularity in recent years, and has led to impressive results in a plethora of fields, ranging from geometry and topology to algebra, logic and type theory. Nonetheless, this approach is not without its limitations: low-dimensional notions of higher category, the formal theory of ∞-groupoids described by homotopy type theory, and the intuitive, informal higher category theory used in practice, are arguably *globular* as opposed to *simplicial*, and tend to be based on explicit operations as opposed to implicit operations, which are merely required to exist because of filling conditions. The *algebraic* approach to higher category theory based on globular operads addresses these issues but there are large gaps in this area, and key notions which have yet to be defined. Consequently, while one can typically take a result of 1-category theory, and then adapt it in order to obtain a result about topological $(\infty, 1)$-categories, this is not currently possible for algebraic higher categories. This thesis aims to build a bridge connecting algebraic and topological models of higher category, as well as approaches based on type theory. In developing this framework, we present the beginnings of a semantics for *directed homotopy type theory*, and we generalize the Batanin-Leinster definition of algebraic higher categories to a definition of higher functors, modules, and transformations.

Our central objects of study are a many-objects generalization of globular operads that we refer to as *globular multicategories*. A guiding intuition is that a globular multicategory can be seen as a *theory of higher categories*. A typical globular multicategory comes with "higher category-like" objects, together with modules between these objects, and arrows between modules. Our hope is that globular multicategories will prove to be a useful tool in model-independent *formal higher category theory*, and we believe that all these data are fundamental for this purpose.

1.1 Overview

We are motivated by two related questions:

1. How can we organize the data needed to reason about higher categories?
2. How can we compare different models of higher category theory?

These questions are closely related since any comparison between different models inevitably organizes the data of these models in some way. Furthermore, we tend to need more and more data about each model, in order to make more and more thorough comparisons. This chapter introduces these questions, and covers two important approaches to higher category theory. We discuss how each of these questions motivates the study of globular multicategories.

1.2 Background

1.2.1 Formal Higher Category Theory

Category theory provides an abstract framework in which to study much of "set-based" mathematics. Familiar constructions such as products and quotients can be characterized via universal properties which determine these notions up to isomorphism. Once a definition has been phrased in such a manner, it is then straightforward to examine models based on categories other than Set. Indeed, topos theory allows one to study the very notion of "set" category-theoretically. Thus, category theory provides a *language* to describe constructions in "set-based" mathematics.

However, category theory itself exhibits structure which cannot be described well using just this language. For example, the notions of equivalence, adjunction and monad all involve natural transformations, the 2-cells of the 2-category of categories. *Formal category theory* aims to describe an abstract language for doing category theory. This language should be abstract enough to reason about "category-like" structures beyond just ordinary categories. For example, many key theorems about categories have analogues about enriched categories and internal categories. We would like all the different versions of each theorem to be instances of a single more general theorem when stated in the right way.

At first, it might seem that these goals can be achieved by reasoning inside a 2-category. Much can be done in this setting. See for example the description of monads and their algebras in [50]. However, some problems, such as describing *pointwise* Kan extensions, depend on a notion of *representability* which cannot be described using just the data of a 2-category. In order to talk about representability a notion of *profunctor* is needed. (In the case of standard categories, profunctors are functors of the form $A^{\mathrm{op}} \times B \to \mathrm{Set}$.) Frameworks for formal category theory include:

- Yoneda Structures (see [52]) : These equip a 2-category with internal Yoneda embeddings.
- Proarrow Equipments (see [58, 59]) : These describe the key properties of the canonical 2-functor $\mathrm{Cat} \to \mathrm{Prof}$ sending $f \mapsto \mathrm{Hom}(\mathrm{id}, f)$. These can also be viewed as double categories with extra structure relating vertical cells ("functors") to horizontal cells ("profunctors") (see [48]).
- Virtual Equipments (see [17]): These generalize proarrow equipments by dropping the requirement that profunctors be composable. Typically, the composite of two profunctors F and G will be a coend in the following sort of form.

  $$\int^{x} F(-, x) \otimes G(x, -)$$

  Virtual equipments can be used to reason about situations where these colimits need not exist or behave well. For example, enrichment in (not necessarily cocomplete) monoidal categories works perfectly well in this setting. Furthermore, virtual equipments allow a number of results to be stated without technical conditions (see [17]).

Finally, if we start with a 2-category with sufficient structure, then we should be able to construct one of these settings for formal category theory. For example, it is shown in [57] that given a 2-topos (a 2-category which behaves like the 2-category of categories), there is an associated Yoneda structure.

Of course, 3-categories such as the 3-category of 2-categories contain extra structure that necessitates an even more expressive language. Hence, we expect the search for a good language for higher categorical mathematics to continue into higher dimensions. Ultimately, we would like to have a language for n-categories for all $n \leq \omega$. However, as n increases, the amount of data involved in definitions tends to increase rapidly.

We will see in this thesis that various collections of "higher category-like" objects can be organized into globular multicategories. Moreover, these globular multicategories frequently come with the extra structure of homomorphism types. Since globular multicategories are a categorification of virtual double categories, it seems reasonable to think that they will provide a natural environment in which to study formal higher category theory.

1.2.2 Topological Higher Categories

Recently a family of models of higher categories which we will refer to as *topological higher categories* has become popular. Given a topological space, we would like there to be an associated fundamental ∞-groupoid of points, paths, homotopies, homotopies between homotopies, etc. In fact, it is considered desirable that this fundamental ∞-groupoid functor be an equivalence of $(\infty, 1)$-categories between topological spaces (up to weak homotopy equivalence) and ∞-groupoids. This is the *homotopy hypothesis*. This correspondence suggests that we could define ∞-groupoids to be topological spaces and study them in an equivalence-invariant way using tools from homotopy theory. Simplicial sets provide a well-known combinatorial model of topological spaces. Hence, we identify ∞-groupoids with simplicial sets (up to weak homotopy equivalence), or more specifically with Kan complexes, the fibrant objects in the standard Kan-Quillen model structure on simplicial sets.

Definition 1.2.2.1. Given an n-simplex Δ, a horn Λ is a simplicial set obtained by discarding an (n-1) cell from the boundary $\partial\Delta$. A *Kan complex* is a simplicial set such that for each horn inclusion $\Lambda \hookrightarrow \Delta$, every commutative diagram

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Delta \arrow[r] \arrow[ur, dotted] & 1
\end{tikzcd}$$

has a filler.

The filling conditions can be thought of as describing both the composition and the inverses of simplicial cells. For example, a composite of two 1-cells f and g corresponds to a filler of the following form:

$$\begin{tikzcd}
\bullet \arrow[r, "g"] \arrow[dr, phantom, "\Rrightarrow"] & \bullet \\
\bullet \arrow[u, "f"] \arrow[ur, dotted, "gf"'] &
\end{tikzcd}$$

Building on these observations we are led to a number of (equivalent) models of (∞,1)-category including:

- Categories enriched in simplicial sets (see for instance [11])
- Quasicategories (see for instance [35]): these are just simplicial sets with slightly fewer filling conditions than Kan complexes so that we can have non-invertible 1-cells.
- Complete Segal Spaces (see for instance [38]): these are intuitively categories internal to simplicial sets whose two notions of equivalence (coming from the category structure and the simplicial set of 0-cells) agree.

There are a number of ways of building on these topological models to obtain categories of (∞, n)-categories for $n \geq 1$. The resulting quasicategories of (∞, n)-categories are all equivalent and are characterized by a few simple axioms in [5].

Importantly, key notions in 1-category theory have been transported to study topological $(\infty, 1)$-categories (see [35]). As a result, a useful theory of $(\infty, 1)$-categories has been developed. Riehl and Verity have developed category theory in a model-independent manner using certain structured simplicially-enriched categories known as ∞-cosmoi (see [41–46]). Their approach allows results such as the Yoneda Lemma to be proved for a large class of models, including topological (∞, n)-categories. Furthermore, they show that much can be done by studying an associated virtual equipments of modules.

Another particularly simple model in this family is given by relative categories.

Definition 1.2.2.2 ([4]). A *relative category* is a category $\mathcal{C}$ together with a subcategory $\mathcal{W}$ of *weak equivalences* such that $\mathcal{W}$ contains all the objects of $\mathcal{C}$.

Given a relative category $(\mathcal{C}, \mathcal{W})$, there exists a *localization* $\mathcal{C}[\mathcal{W}^{-1}]$. This is the $(\infty, 1)$-category obtained from $\mathcal{C}$ by inverting (in a weak sense) the morphisms of $\mathcal{W}$. In fact, $(\infty, 1)$-categories can be identified with relative categories in this way (see [4]). Notably, many important categories in algebraic topology and homological algebra come equipped with natural notions of weak equivalences. Examples include:

- weak homotopy equivalences between topological spaces,
- quasi-isomorphisms between chain complexes.

Frequently, relative categories underlie more structured categories such as *model categories*. These come with notions of *fibration* and/or *cofibration* that can be thought of as inclusions and projections which interact well with weak equivalences.

Whilst in some sense the weak equivalences are all that is needed in order to determine the relevant higher category, this extra structure is often useful for equivalence-invariant *computation.* Homotopical models of intentional type theory can be seen as an embodiment of this idea. (See for example [49].) From this perspective, fibrations can be seen as *display maps* which tell us how to model dependent types. A key feature of such models is that diagonal maps have factorizations

$$A \longrightarrow A^I \twoheadrightarrow A \times A$$

where the right-hand arrow is a fibration. If the object A is seen as a type in a type theory, then the path space object A^I models the *identity type* of A. In this way, we can obtain a model of dependent type theory with intensional identity types from any "sufficiently nice" $(\infty, 1)$-category. Awodey and Warren first made this connection in [3]. Thus, *homotopy type theory* provides a synthetic language for reasoning in an equivalence-invariant fashion about ∞-groupoid-like objects (see [53]). Moreover, a guiding intuition is that there should be a higher adjunction between a suitable category of homotopy type theories and a category of nice $(\infty, 1)$-categories. Much work has been done in this direction, although some work still remains. (See for example [27] and [28].) A recent approach to extending homotopy type theory for reasoning about more general (∞, n)-categories is [40].

Conceptually, the ∞-groupoids described by homotopy type theory are quite different from the topological models of higher categories. The tower of identity types gives types a *globular* structure. However, topological models tend to be simplicial. For example, a common way to describe the Hom object of a quasi-category is to change model, view it as a simplicially-enriched category, and then look at the corresponding simplicial set of 1-cells. Thus, fundamental operations of homotopy type theory and of the informal reasoning that people use in practice to reason about higher categories are arguably not captured very well by topological models.

In this thesis, we will show that models of dependent type theory with identity types induce globular multicategories with homomorphism types. Thus, our results can be seen as a first step towards a globular semantics for homotopy type theory. We show how this framework suggests directed generalizations of more familiar models of dependent type theory.

1.2.3 Algebraic Higher Categories

A different class of models of higher category, the globular models of higher category in the style of Batanin, can be constructed quite directly from intensional type theories.

(See [6] for Batanin's original definition of higher categories using globular operads, and see [30] for a popular variation due to Leinster. For constructions of these globular models starting from an intensional type theory, see [34, 54].) We build on these results by describing methods to construct algebraic higher categories given a globular multicategory with homomorphism types.

Algebraic models consist of operations sending pasting diagrams to cells. Since these operations are specified explicitly, we call these models *algebraic*. This seems to be a good match for the constructive style of type theory, which asks for proof witnesses. Indeed, Finster and Mimram [21] have shown how an algebraic notion of weak ω-category in this family of models can be described using a type-theoretic calculus. Another key feature of these models is that they are inherently *directed*: an ω-groupoid is just a ω-category whose n-cells are equivalences for $n \geq 1$. Thus, algebraic higher categories seem much closer in spirit to the usual low-dimensional notions of higher category, as well as informal notions of higher category.

However, the usefulness of these models is severely limited by the lack of definitions of key notions. For example, it remains an open problem to describe the weak ω-category of weak ω-categories in this setting, or even to describe a notion of composition along k-cells for $k > 0$. (See [23] for one proposed definition of k-cells (higher transformations) and composition along 0-cells.) We discuss how the theory of globular multicategories suggests another possible family of definitions for these objects.

1.3 Summary of this Thesis

Chapter 2 provides a detailed introduction to globular multicategories, and introduces the fundamental notions and notations that permeate this thesis. We provide an introduction to the basic structures on which the notion of globular multicategory depends: pasting diagrams, and the free strict ω-category monad. We then define globular multicategories themselves. The presented definition is not new but our choice of notation is novel. We adopt type-theoretic terminology for the data in each globular multicategory; we believe that this approach greatly facilitates reasoning about these data. We then assemble a collection of examples and constructions, both novel and previously known, that undergirds and illustrates the remaining chapters.

The Hom-profunctor plays a fundamental role in category theory, and higher dimensional generalizations are needed to reason about higher categories. Chapter 3 describes how a globular multicategory can be equipped with *strict homomorphism* *types* that play this role. We show how various sorts of strict n-categories can be organized into globular multicategories with strict homomorphism types. Finally, we introduce a *higher dimensional modules construction* that constructs globular multicategories with strict homomorphism types, and we show how this construction is intimately linked to iterated strict enrichment.

Many interesting higher dimensional examples do not satisfy the identities required for the construction of strict homomorphism types, although they do satisfy these identities up to a suitable notion of equivalence. In order to solve this problem, Chapter 4 introduces two weak variants of homomorphism types in a globular multicategory; *fibrational homomorphism types* behave like a directed version of identity types in a type theory, while *weak homomorphism types* behave like directed path types (propositional identity types) in a type theory. We show how type theories with identity types (or path types) induce corresponding globular multicategories with fibrational (or weak) homomorphism types. In fact, our results rely on two-sided notions of factorization, generalizing the familiar one-dimensional factorization systems that provide semantics of undirected homotopy type theories. In this way, we believe that globular multicategories with fibrational (or weak) homomorphism types provide a natural environment for the study of the semantics of future directed homotopy type theories.

The previous chapters show how various higher categorical structures induce globular multicategories with homomorphism types. Chapter 5 contains results in the other direction: given a globular multicategory with homomorphism types, we construct higher categorical objects.

1.4 Summary of Contributions

Here we highlight the contributions of this thesis.

Chapter 2 introduces the vertical construction, and the families construction for globular multicategories. It describes a universal property for families constructions. We also introduce a novel type-theoretic notation here, which simplifies previously known notions. For example, using this notation, the defining property of the 2-cells of GlobMult closely resembles the usual notion of natural transformation.

Chapter 3 is almost entirely novel. We introduce strict homomorphism types, and construct a number of examples. We construct level-wise strict higher modules functors and prove their universal properties. We then use these level-wise functors to construct more complicated higher modules functors. We show how iterated enrichment can be understood as the result of applying a families construction and then a modules construction. In particular, this implies that the globular multicategory of modules in SpanSet can be seen as the collection of strict ω-categories, strict profunctors between them, and strict transformations between these objects.

Chapter 4 is also almost entirely novel. We introduce globular multicategories with pre-homomorphism types, fibrational homomorphism types, and weak homomorphism types. We show how these definitions are all characterized by certain representing properties of reflexivity substitutions. We give new notions of category with pre-homomorphism types, fibrational homomorphism types, and weak homomorphism types. We show how these notions generalize categories with identity types and path types.

Chapter 5 is a straightforward application of the new tools developed in the previous chapters. We discuss how the data of categories with strict and fibrational homomorphism types can be equipped with higher categorical structure.

Combining these results, we significantly develop the theory of globular multicategories. We obtain new constructions of algebraic higher categorical structures from topological higher categories, and we do this in a manner that suggests a new approach to the semantics of dependent type theory with (directed) identity types.

1.5 Related Work

Here we highlight the most significant influences of this work, as well as some closely related ideas in the literature.

One-dimensional globular multicategories are virtual double categories. This case is by far the best understood case in the literature and is thoroughly developed in [17]. This case, while far simpler and more familiar than the general case, is already sufficiently rich to illuminate many of the key constructions and definitions of this thesis. In fact, many of our new results can be viewed as higher dimensional generalizations of results about virtual double categories. In particular, the monoids and modules construction on virtual double categories is the one-dimensional version of the higher modules construction introduced in this thesis. This construction was first introduced by Leinster [31], and a universal property is described in [17].

The idea of equipping globular multicategories with homomorphism types is almost present in [54]. Here, monoidal globular categories play an intermediate step in the construction of ω-groupoids from dependent type theories with identity types.

This approach to the semantics of type theory greatly influenced Chapter 4. Many of our results on type theories are also closely related to results of Lumsdaine [34], although our focus is more on the algebraic structures giving type theories semantics, than on syntactic constructions.

Our approach to algebraic higher categorical structures builds on Batanin and Leinster's approach to higher categories [6, 30] as well as Garner's homotopical interpretation of Leinster's approach [22]. The weak notions of higher functor that we propose are closely related to the coherence results for functors between 2-categories and 3-categories described by Gurski [24].

The idea of weakening higher transformations using generalized operads and algebraic contractions has been studied independently by Kachour [25, 26]. Kachour's work on this topic has gone through numerous revisions, and various similar approaches are described.

Notably absent in this thesis is a comparison with the ∞-cosmoi of Riehl and Verity (see [41–46]). Given that they construct virtual equipments of modules, this seems like a promising future direction.

1.6 Future Work

Another goal for future work is to clarify the type-theoretic nature of globular multicategories. Each globular multicategory should induce a model of dependent type theory in the form of a category with families (see [19]), or equivalently a natural model of type theory (see [2]). There should be a close correspondence between the types, terms, contexts, etc. of a globular multicategory and the synonymous objects in the associated category with families. Hence, this approach could give a precise justification for the type-theoretic terminology used throughout this thesis. Moreover, results about globular multicategories could be effectively translated into results about dependent type theories and vice versa. For example, when a globular multicategory is representable, the corresponding type theory should support certain Σ-types; when a globular multicategory has homomorphism types, the corresponding type theory should support identity types. It would be very interesting to study the analogue of type-theoretic universes in the setting of globular multicategories. Since the objects of globular multicategories with homomorphism types behave like collections of higher categories, homomorphism types of the universe could provide a great deal of information about the globular multicategory of all categories and the globular multicategory of all globular multicategories.

Chapter 2

Globular Multicategories

In this chapter, we introduce our central objects of study: globular multicategories. A globular multicategory amounts to a globular set of *types*, together with composable collections of *terms*; terms are arrows sending pasting diagrams of types to other types. Our first goal is to make this definition precise. We then introduce a variety of examples and constructions that we will use in later chapters. Much of this material can already be found throughout the literature. Here we organise this material, and supplement it with a handful of new results.

2.1 Preliminary Notions

In this section, we lay the groundwork for the definition of globular multicategories. We provide a self-contained introduction to the categories of globes, globular sets, and pasting diagrams. Finally, we examine how the free strict ω-category monad $\mathbf{T}$ can be succinctly described using pasting diagrams.

2.1.1 Globes

We first examine what is meant by the term *globular*. In general, a globular object is parametrised by the category of *globes*, whose objects are points, arrows, arrows between arrows, etc.

Definition 2.1.1.1. The category of *globes*, $\mathbb{G} = \mathbb{G}_\omega$, is freely generated by the morphisms

$$\begin{tikzcd}
0 \arrow[r, shift left, "\sigma_{0,1}"] \arrow[r, shift right, "\tau_{0,1}"'] & 1 \arrow[r, shift left, "\sigma_{1,2}"] \arrow[r, shift right, "\tau_{1,2}"'] & \cdots \arrow[r, shift left, "\sigma_{k-1,k}"] \arrow[r, shift right, "\tau_{k-1,k}"'] & k \arrow[r, shift left, "\sigma_{k,k+1}"] \arrow[r, shift right, "\tau_{k,k+1}"'] & \cdots
\end{tikzcd}$$

subject to the *globularity conditions*:

$$\sigma_{k+1,k+2} \circ \sigma_{k,k+1} = \tau_{k+1,k+2} \circ \sigma_{k,k+1},$$

$$\sigma_{k+1,k+2} \circ \tau_{k,k+1} = \tau_{k+1,k+2} \circ \tau_{k,k+1}.$$

For each n, we refer to the object n as the *n-globe*. We depict the 0-globe by a point, and for $n > 0$, we depict the n-globe as an arrow between $(n-1)$-globes. Thus, the 0-globe, 1-globe, 2-globe and 3-globe are depicted as follows:

$$\begin{tikzcd}
\bullet & \bullet \arrow[r] & \bullet & \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D] & \bullet \arrow[r, bend left, ""{name=U2, below}] \arrow[r, bend right, ""{name=D2}] & \bullet \arrow[Rightarrow, from=U2, to=D2, shift right=2, bend right, ""{name=L}] \arrow[Rightarrow, from=U2, to=D2, shift left=2, bend left, ""{name=R, left}] \arrow[Rrightarrow, from=L, to=R]
\end{tikzcd}$$

The morphisms of the globe category can be seen picking out the *sources* and *targets* of these arrows. The globularity condition then says that "the source of the source is the source of the target" and "the target of the source is the target of the target". It follows that for each $k < n$, there are exactly two arrows $k \to n$ in $\mathbb{G}$. We denote these arrows, which factor through $\sigma_{k,k+1}$ and $\tau_{k,k+1}$ respectively, by

$$\sigma_{k,n} : k \longrightarrow n, \qquad \tau_{k,n} : k \longrightarrow n.$$

Note that n is typically clear from the context, and so we also write $\sigma_k, \tau_k : k \to n$. We define $\mathbb{G}_n$ to be the full subcategory of $\mathbb{G}$ on the objects $0, \dots, n$. For each $0 \leq n \leq \omega$, an *n-globular object* in a category $\mathcal{C}$ is a functor $A : \mathbb{G}_n^{\mathrm{op}} \to \mathcal{C}$. We denote the image of $\sigma_k, \tau_k : k \to n$ under such a functor by

$$A(n) \overset{s_k}{\underset{t_k}{\rightrightarrows}} A(k),$$

and refer to these morphisms as the *k-source* and *k-target* morphisms. We will also write s, t for the arrows $s_{l-1}, t_{l-1} : A(l) \to A(l-1)$ respectively, and refer to these maps as the *source* and *target* morphisms respectively.

Definition 2.1.1.2. An n-globular object in Set is called an *n-globular set*, while an n-globular object in Cat is called an *n-globular category*. In these cases, we refer to the elements of $A(k)$ as *k-cells*. We depict k-cells by labelled k-globes. Thus, the representable globular sets corresponding to the 0, 1 and 2-globe could be depicted as follows:

$$\begin{tikzcd}
A & A \arrow[r, "f"] & B & A \arrow[r, bend left, "f", ""{name=U, below}] \arrow[r, bend right, "g"', ""{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\phi"] & A \arrow[r, bend left, "f", ""{name=U2, below}] \arrow[r, bend right, "g"', ""{name=D2}] & B \arrow[Rightarrow, from=U2, to=D2, shift right=2, bend right, "\phi"', ""{name=L}] \arrow[Rightarrow, from=U2, to=D2, shift left=2, bend left, "\psi", ""{name=R, left}] \arrow[Rrightarrow, from=L, to=R, "\theta"]
\end{tikzcd}$$

The following diagrams depict more complicated examples of globular sets:

$$\begin{tikzcd}
A \arrow[r, bend left=60, "f", ""{name=U, below}] \arrow[r, "g" description, ""{name=M}] \arrow[r, bend right=60, "g"', ""{name=D}] & B \arrow[Rightarrow, from=U, to=M, "\phi"] \arrow[Rightarrow, from=M, to=D, "\psi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
{} & D \arrow[loop left, "m"] \arrow[d, "k"] & {} \\
A \arrow[ur, "l"] \arrow[r, bend left, "f", ""{name=U, below}] \arrow[r, bend right, "g"', ""{name=D}] & B \arrow[r, "i" description, ""{name=I}] \arrow[r, bend right, "j"', ""{name=J}] & C \arrow[l, bend right, "h"'] \arrow[Rightarrow, from=U, to=D, "\phi"] \arrow[Rightarrow, from=I, to=J, "\psi"]
\end{tikzcd}$$

The source and target maps pick out the sources and targets of the depicted arrows. For example, in the last two diagrams above, we have that $s\phi = f$ and $t_0\phi = B$. We say that two n-cells a, b are *parallel* when $sa = sb$ and $ta = tb$. For example, in the above right diagram the 1-cells i and j are parallel, while i and h are not parallel. The globularity condition for globular sets amounts to the requirement that the source and target of any n-cell be parallel.

Remark 2.1.1.3. There are obvious fully faithful inclusions,

$$\mathbb{G}_0 \hookrightarrow \mathbb{G}_1 \hookrightarrow \cdots \hookrightarrow \mathbb{G}_\omega.$$

Let $\mathcal{C}$ be any category. Then composition with these inclusions induces *truncation* functors

$$[\mathbb{G}_0^{\mathrm{op}}, \mathcal{C}] \xleftarrow{\mathrm{tr}_0} [\mathbb{G}_1^{\mathrm{op}}, \mathcal{C}] \xleftarrow{\mathrm{tr}_1} \cdots \longleftarrow [\mathbb{G}_\omega^{\mathrm{op}}, \mathcal{C}]$$

When $\mathcal{C} = \mathrm{Set}$, the truncation functor tr_k forgets all n-cells for $n > k$. When $\mathcal{C}$ has an initial object, each of these functors has a left adjoint L_{tr_k}. Suppose that $k < n$, and that A is a k-globular object in $\mathcal{C}$. Then, we have that

$$L_{\mathrm{tr}_k}(A)(i) = \begin{cases} A(i) & \text{if } i \leq k \\ \emptyset & \text{if } i > k \end{cases}$$

This left adjoint is fully faithful, and we frequently identify a k-globular set, A, with the n-globular set $L_{\mathrm{tr}_k}(A)$. Indeed, we will particularly focus on ω-globular sets since results about ω-globular sets can typically be transformed into results about n-globular sets for all n by taking truncations. We refer to ω-globular sets simply as *globular sets.* We define the *dimension* of an globular set A by

$$\dim A = \max\{n \mid A(n) \neq \emptyset\}.$$

For example, each of the examples of globular sets depicted by a diagram above has dimension at most 3. It follows that a globular set A is in the image of $L_{\mathrm{tr}_n}(A')$ for some n-globular set A' if and only if $\dim A = n$.

2.1.2 Globular Pasting Diagrams

The *globular pasting diagrams* are an important class of globular sets. Each globular pasting diagram describes a notion of composition in a higher category. For example, there is a globular pasting diagram consisting of a pair of composable 1-cells:

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{2.1.2.a}$$

This diagram describes the usual notion of composition in a category: whenever we have arrows f and g as above, there is a canonical composite $g \circ f$. On the other hand, the globular set

$$A \xrightarrow{f} B \xleftarrow{g} C$$

is not a globular pasting diagram, since we do not in general expect such diagrams to induce any sort of composition in a category. There are a number of equivalent ways of describing globular pasting diagrams in the literature. See for example [6, 30, 51] where these objects are called *n-stage trees*, *globular cardinals*, and *pasting diagrams*. We present two different descriptions here.

On the one hand, the collection of pasting diagrams can be inductively defined, starting with the globes, and then defining new pasting diagrams by "gluing the target of one diagram to the source of another".

Definition 2.1.2.1. For each $n \geq 0$, we define the following data:

- A set $\mathbf{pd}^{\mathrm{br}}(n)$ whose elements we refer to as *bracketed n-pasting diagrams*,
- For each $\pi \in \mathbf{pd}^{\mathrm{br}}(n)$, a globular set $\mathcal{I}\pi$,
- For each $k < n$, and each bracketed n-pasting diagram π, a bracketed k-pasting diagram π_{∂_k} together with *source and target maps*

$$\sigma_{\pi,k} : \mathcal{I}\pi_{\partial_k} \longrightarrow \mathcal{I}\pi, \qquad \tau_{\pi,k} : \mathcal{I}\pi_{\partial_k} \longrightarrow \mathcal{I}\pi$$

satisfying globularity conditions:

$$\sigma_{\pi,j} \circ \sigma_{\pi_{\partial_j},k} = \sigma_{\pi,k} = \tau_{\pi,j} \circ \sigma_{\pi_{\partial_j},k},$$
$$\sigma_{\pi,j} \circ \tau_{\pi_{\partial_j},k} = \tau_{\pi,k} = \tau_{\pi,j} \circ \tau_{\pi_{\partial_j},k}.$$

Firstly, when $n = 0$, there is a unique bracketed 0-pasting diagram $\mathbf{D}^0$ such that $\mathcal{I}\mathbf{D}^0$ is the representable 0-globe. Now suppose that $n > 0$. Then we define the data simultaneously by induction:

- There is a bracketed n-pasting diagram $\mathbf{D}^n$ such that $\mathcal{I}\mathbf{D}^n$ is the representable n-globe. Whenever $k < n$, we define $(\mathbf{D}^n)_{\partial_k} = \mathbf{D}^k$, and we define σ_k and τ_k to be the arrows induced by the corresponding arrows in $\mathbb{G}$.

- Each bracketed $(n-1)$-pasting diagram π induces a bracketed n-pasting diagram π^+ such that $\mathcal{I}\pi^+ = \mathcal{I}\pi$. In this case, we set $\pi^+_{\partial_n} = \pi$; we define the boundary inclusions by

$$\sigma_{\pi^+,k} = \begin{cases} \mathrm{id}_{\mathcal{I}\pi} & \text{if } k = n \\ \sigma_{\pi,k} & \text{if } k < n \end{cases} \qquad \tau_{\pi^+,k} = \begin{cases} \mathrm{id}_{\mathcal{I}\pi} & \text{if } k = n \\ \tau_{\pi,k} & \text{if } k < n \end{cases}$$

  When there is no danger of ambiguity, we will elide this notation and simply refer to π^+ as π.

- Given bracketed n-pasting diagrams π_1, π_2 such that $(\pi_1)_{\partial_k} = (\pi_2)_{\partial_k} = \rho$, there is a bracketed n-pasting diagram denoted $\pi_1 \odot_k \pi_2$. We define $\mathcal{I}(\pi_1 \odot_k \pi_2)$ to be the following pushout:

$$\begin{tikzcd}
\mathcal{I}\rho \arrow[r, "\tau_{\pi_1,k}"] \arrow[d, "\sigma_{\pi_2,k}"'] & \mathcal{I}\pi_1 \arrow[d] \\
\mathcal{I}\pi_2 \arrow[r] & \mathcal{I}(\pi_1 \odot_k \pi_2)
\end{tikzcd}$$

  When $j < k$ we define $(\pi_1 \odot_k \pi_2)_{\partial j} = \rho_j$, and we define $\sigma_{\pi,j}$, $\tau_{\pi,j}$ to be the composites

$$\mathcal{I}\rho_{\partial_j} \xrightarrow{\sigma_{\rho,j}} \mathcal{I}\rho \longrightarrow \mathcal{I}(\pi_1 \odot_k \pi_2), \qquad \mathcal{I}\rho_{\partial_j} \xrightarrow{\tau_{\rho,j}} \mathcal{I}\rho \longrightarrow \mathcal{I}(\pi_1 \odot_k \pi_2).$$

  When $j = k$, we define $(\pi_1 \odot_k \pi_2)_{\partial j} = (\pi_1)_{\partial j} = (\pi_2)_{\partial j} = \rho$, and we define σ_j, τ_j to be the composites

$$\mathcal{I}(\pi_1)_{\partial_j} \xrightarrow{\sigma_{\pi_1,j}} \mathcal{I}\pi_1 \longrightarrow \mathcal{I}(\pi_1 \odot_k \pi_2), \qquad \mathcal{I}(\pi_2)_{\partial_j} \xrightarrow{\tau_{\pi_2,j}} \mathcal{I}\pi_2 \longrightarrow \mathcal{I}(\pi_1 \odot_k \pi_2)$$

  When $j > k$, we define $(\pi_1 \odot_k \pi_2)_{\partial_j} = (\pi_1)_{\partial_j} \odot_k (\pi_2)_{\partial_j}$. The source and target maps are induced by the universal properties of these pushouts. For example, by the globularity conditions, we have that $\sigma_{\pi_2,j} \circ \sigma_{(\pi_2)_{\partial j},k} = \sigma_{\pi_2,k}$ and $\sigma_{\pi_1,j} \circ \tau_{(\pi_1)_{\partial j},k} = \tau_{\pi_1,k}$. Hence, we define $\sigma_{\pi_1 \odot_k \pi_2,j}$ to be the canonical arrow making the following diagram commute:

$$\begin{tikzcd}
\mathcal{I}\rho \arrow[r, "\tau_{(\pi_1)_{\partial j},k}"] \arrow[d, "\sigma_{(\pi_2)_{\partial j},k}"'] & \mathcal{I}(\pi_1)_{\partial j} \arrow[r, "\sigma_{\pi_1,j}"] \arrow[d] & \mathcal{I}\pi_1 \arrow[dd] \\
\mathcal{I}(\pi_2)_{\partial j} \arrow[r] \arrow[d, "\sigma_{\pi_2,j}"'] & \mathcal{I}(\pi_1)_{\partial j} \odot_k \mathcal{I}(\pi_2)_{\partial j} \arrow[dr, dotted, "\sigma_{\pi_1 \odot_k \pi_2,j}"] & \\
\mathcal{I}\pi_2 \arrow[rr] & & I\pi_1 \odot_k \mathcal{I}\pi_2
\end{tikzcd}$$

  These definitions are easily seen to satisfy the globularity conditions.

We view $\mathbf{pd}^{\mathrm{br}}(n)$ as a category by defining an arrow $\pi \to \rho$ in $\mathbf{pd}^{\mathrm{br}}(n)$ to be a map of globular sets $\mathcal{I}\pi \to \mathcal{I}\rho$. We define $\mathbf{pd}^{\mathrm{br}}$ to be the globular set whose n-cells are bracketed n-pasting diagrams; source and targets are defined so that, for all k,

$$s_k\pi = t_k\pi = \pi_{\partial_k}.$$

Given a globular set X, the *category of elements* $\mathrm{el}(X)$ is defined so that:

- An object in $\mathrm{el}(X)$ is a pair (n, x), where $n \in \mathbb{G}$ and $x \in X(n)$.
- Suppose that $f : m \to n$ is an arrow in $\mathbb{G}$, and suppose that we have an object $(n, x) \in \mathrm{el}(X)$. Let $y = X(f)(x)$. Then, there is an arrow $(f, x) : (m, y) \to (n, x)$ in $\mathrm{el}(X)$.
- Composition of arrows in $\mathrm{el}(X)$ comes from $\mathbb{G}$.

It follows that there is a functor $\mathcal{I}^{\mathrm{br}} : \mathrm{el}(\mathbf{pd}^{\mathrm{br}}) \to \mathbb{G}\text{-Set}$ defined by

$$\mathcal{I}^{\mathrm{br}}(n,\pi) = \mathcal{I}(\pi), \qquad \mathcal{I}^{\mathrm{br}}(\sigma_k,\pi) = \sigma_{\pi,k}, \qquad \mathcal{I}^{\mathrm{br}}(\tau_k,\pi) = \tau_{\pi,k}.$$

Example 2.1.2.2. The diagram (2.1.2.a) is the result of pasting two 1-cells along a shared 0-cell; we have the following pushout diagram:

$$\begin{tikzcd}
 & A \arrow[r, "f"] & B \arrow[dr, hook] & & & \\
B \arrow[ur, hook] \arrow[dr, hook] & & & A \arrow[r, "f"] & B \arrow[r, "g"] & C \\
 & B \arrow[r, "g"] & C \arrow[ur, hook] & & &
\end{tikzcd}$$

This exhibits diagram (2.1.2.a) as $\mathcal{I}(\mathbf{D}^1 \odot_0 \mathbf{D}^1)$.

Example 2.1.2.3. A 2-cell and a 1-cell can be pasted along a shared 0-cell in two ways:

$$\begin{tikzcd}
 & A \arrow[r, "f"] & B \arrow[dr, hook] & & & \\
B \arrow[ur, hook] \arrow[dr, hook] & & & A \arrow[r, "f"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\Downarrow\psi"] & C \\
 & B \arrow[r, bend left, "g"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\Downarrow\psi"] & C \arrow[ur, hook] & & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[hook, drr] & & & & \\
B \arrow[hook, urr] \arrow[hook, drr] & & & & & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, "g"] & C \\
& & B \arrow[r, "g"] & C \arrow[hook, urr] & & & &
\end{tikzcd}$$

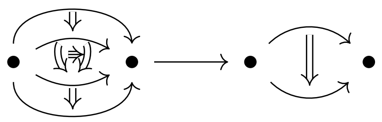
The resulting pushouts, denoted $\mathcal{I}(\mathbf{D}^1 \odot_0 \mathbf{D}^2)$ and $\mathcal{I}(\mathbf{D}^2 \odot_0 \mathbf{D}^1)$, correspond to whiskering operations in a 2-category.

Example 2.1.2.4. Combining the previous two examples, we obtain the following diagram:

$$\begin{tikzcd}
& & & & A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, "g"] & C \arrow[hook, drr] & & & & \\
A \arrow[r, "h"] & B \arrow[r, "g"] & C \arrow[hook, urr] \arrow[hook, drr] & & & & & & A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\Downarrow\psi"] & C \\
& & & & A \arrow[r, "h"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\psi\Downarrow"] & C \arrow[hook, urr] & & & &
\end{tikzcd}$$

The right-hand side is clearly $\mathcal{I}(\mathbf{D}^2 \odot_0 \mathbf{D}^2)$. Hence, this pushout diagram tells us that $(\mathbf{D}^2 \odot_0 \mathbf{D}^1) \odot_{\mathbf{D}^1} (\mathbf{D}^1 \odot_0 \mathbf{D}^2) \cong (\mathbf{D}^2 \odot_0 \mathbf{D}^2)$; this corresponds to the well known fact that in a 2-category horizontal composition can be defined using a combination of left-whiskering, right-whiskering and vertical composition.

Example 2.1.2.5. The following diagram depicts the globular set associated to the 3-pasting diagram $(\mathbf{D}^2 \odot_1 (\mathbf{D}^3 \odot_1 \mathbf{D}^2)) \odot_0 (\mathbf{D}^1 \odot_0 \mathbf{D}^2)$ with labels omitted:



Example 2.1.2.6. Familiar laws of composition in higher categories follow from the commutativity of colimits. For example, for each $k < n$, we have the following *associativity and unit laws*:

$$\pi \odot_k \mathbf{D}^k \cong \pi \cong \mathbf{D}^k \odot_k \pi, \qquad (o \odot_k \pi) \odot_k \rho \cong o \odot_k (\pi \odot_k \rho),$$

We will henceforth denote the composite $\pi_1 \odot_k (\pi_2 \odot_k \cdots \odot_k \pi_l) \cdots)$ by $\pi_1 \odot_k \pi_2 \odot_k \cdots \odot_k \pi_l$. Commutativity of colimits also implies that, for each $i, j < n$, we have the following *interchange law*:

$$(\pi \odot_j \pi') \odot_i (\rho \odot_j \rho') \cong (\pi \odot_i \rho) \odot_j (\pi' \odot_i \rho').$$

These examples serve to illustrate that bracketed pasting diagrams have an intuitive graphical flavour. However, Example 2.1.2.4 exhibits one shortcoming of this approach: there are non-trivial relations between the operations $\odot_0, \odot_1, \ldots$, and this makes constructions using this definition more involved. For example, it not a priori clear that the operations $\odot_0, \odot_1, \ldots$ respect isomorphisms in $\mathbf{pd}^{\mathrm{br}}(n)$. Another way to understand this difficulty is to note that the pushouts involved in the definition of $\mathbf{pd}^{\mathrm{br}}(n)$ implicitly involve computing certain quotients. In contrast, the following construction avoids this issue by defining globular pasting diagrams to be simple inductively defined objects without using any quotients.

Definition 2.1.2.7. Let $\star = ()$ be the empty list. For each $n \geq 0$, we define the set $\mathbf{pd}(n)$ of *n-pasting diagrams* by induction on n:

- When $n = 0$, we define $\mathbf{pd}(0) = \{\star\}$,
- When $n > 0$, we define $\mathbf{pd}(n)$ to be the set of lists $(\pi_1, \pi_2, \ldots, \pi_l)$ such that $l \geq 0$ and $\pi_i \in \mathbf{pd}(n-1)$.

We view $\mathbf{pd}$ as a globular set by defining, for each $\pi \in \mathbf{pd}(n)$,

$$s\pi = \star, \qquad t\pi = \star,$$

when $n = 1$, and

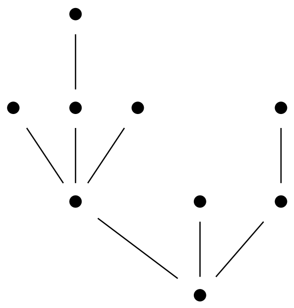
$$s(\pi_1, \ldots, \pi_l) = (s\pi_1, \ldots, s\pi_l), \qquad t(\pi_1, \ldots, \pi_l) = (t\pi_1, \ldots, t\pi_l),$$

when $n > 1$.

Remark 2.1.2.8. These lists are often viewed as *trees*. Given an element π in $\mathbf{pd}(n)$, we define a finite tree $\mathrm{Tr}\,\pi$ of depth at most n, by induction on n as follows:

- When $n = 0$, we define $\mathrm{Tr}\,\star$ to be the tree with a unique vertex.
- When $n > 0$, and $\pi = (\pi_1, \pi_2, \ldots, \pi_l) \in \mathbf{pd}(n)$, we define $\mathrm{Tr}\,\pi$ to be the tree whose root has a child π_i for each $1 \leq i \leq l$, such that the subtree whose root is π_i is $\mathrm{Tr}\,\pi_i$.

For example, the 3-pasting diagram $((\star, (\star), \star), \star, (\star))$ corresponds to the following tree:



Example 2.1.2.9. Each n-pasting diagram π can be seen as an $(n+1)$-pasting diagram π^+. This amounts to the fact that every tree of depth at most n has depth at most $(n+1)$.

Definition 2.1.2.10. The arrows of $\mathbf{pd}(n)$ are defined inductively. When $n = 0$, there is a unique arrow $\mathrm{id}_\star : \star \to \star$. Suppose that $n > 0$, and that $\pi = (\pi_1, \ldots, \pi_l)$ and $\rho = (\rho_1, \ldots, \rho_m)$ are elements of $\mathbf{pd}(m)$. Then to give an arrow $f : \pi \to \rho$ is to choose

- A natural number $\mathbf{j}_f \geq 0$ such that $\mathbf{j}_f + l \leq m$. We view this as an embedding of lists of length l into lists of length m.
- For each $1 \leq i \leq l$, an arrow $f_i : \pi_i \to \rho_{\mathbf{j}_f + i}$ in $\mathbf{pd}(n-1)$.

Composition is defined by

$$\mathbf{j}_{g \circ f} = \mathbf{j}_g + \mathbf{j}_f, \qquad f_{(g \circ f)_i} = g_{\mathbf{j}_f + i} \circ f_i.$$

The identity arrows are defined by

$$\mathbf{j}_{\mathrm{id}_\pi} = 0, \qquad (\mathrm{id}_\pi)_i = \mathrm{id}_{\pi_i}.$$

These data suffice to make $\mathbf{pd}(n)$ a category. Suppose that $n > 0$, and that $f : \pi \to \rho$ is an arrow of $\mathbf{pd}(n)$. We define $sf : s\pi \to s\rho$ and $tf : t\pi \to t\rho$ by induction on n. When $n = 1$, we define

$$sf = tf = \mathrm{id}_\star.$$

When $n > 1$, we define

$$\mathbf{j}_{sf} = \mathbf{j}_f, \qquad (sf)_i = s(f_i),$$
$$\mathbf{j}_{tf} = \mathbf{j}_f, \qquad (tf)_i = t(f_i).$$

It is easily seen that taking sources and targets is functorial and satisfies the globularity condition. Hence, we have equipped $\mathbf{pd}$ with the structure of a globular category.

Example 2.1.2.11. The assignment $-^{+} : \mathbf{pd}(n) \to \mathbf{pd}(n+1)$ is the objects part of a functor. Given $f : \pi \to \rho$, we set

$$\mathbf{j}_{f^{+}} = 0, \qquad (f^{+})_i = f_i.$$

Example 2.1.2.12. Suppose that $n > 0$, and that $\pi = (\pi_1, \ldots, \pi_l)$ is an n-pasting diagram. Then, we define *source and target inclusions*

$$\sigma_\pi : \pi_\partial^{+} \longrightarrow \pi, \qquad \tau_\pi : \pi_\partial^{+} \longrightarrow \pi$$

by induction on n. When $n = 1$, we define

$$\mathbf{j}_{\sigma_\pi} = 0, \qquad \mathbf{j}_{\tau_\pi} = l.$$

This completes the definition in this case, since $\star^{+}$ has length 0. When $n > 1$, we define

$$\mathbf{j}_{\sigma_\pi} = 0, \qquad \mathbf{j}_{\tau_\pi} = 0,$$

$$(\sigma_\pi)_i = \sigma_{\pi_i}, \qquad (\tau_\pi)_i = \tau_{\pi_i}.$$

It is easily verified that the source and target inclusions satisfy globularity conditions.

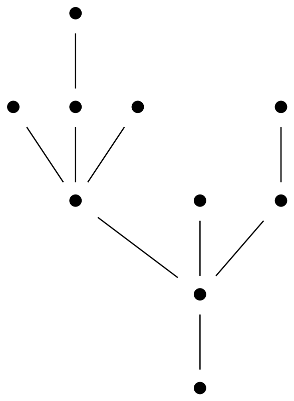
Definition 2.1.2.13. We define the *suspension functor* $\Sigma : \mathbf{pd}(n) \to \mathbf{pd}(n+1)$ on objects by

$$\Sigma\pi = (\pi),$$

and on arrows by

$$\mathbf{j}_{\Sigma f} = 1, \qquad (\Sigma f)_1 = f.$$

Thus, Σ grows a tree π by adding a new root vertex whose only child is the root of π. For example, when π is the tree in Remark 2.1.2.8, we have that $\Sigma\pi$ is the following tree:



Example 2.1.2.14. Suppose that $n > 0$, and suppose that $\pi_1, \pi_2 \in \mathbf{pd}(n)$ where $\pi_1 = (\pi_{1,1}, \ldots, \pi_{1,l})$ and $\pi_2 = (\pi_{2,1}, \ldots, \pi_{2,m})$. Then we define the n-pasting diagram $\pi_1 \odot_0 \pi_2$ by list concatenation:

$$\pi_1 \odot_0 \pi_2 = (\pi_{1,1}, \ldots, \pi_{1,l}, \pi_{2,1}, \ldots, \pi_{2,m}).$$

Suppose that $f : \pi_1 \to \rho_1$ and $g : \pi_2 \to \rho_2$ are arrows in $\mathbf{pd}'(n)$. Then we define $f \odot_0 g : \pi_1 \odot_0 \pi_2 \to \rho_1 \odot_0 \rho_2$ by

$$\mathbf{j}_{f \odot_0 g} = \mathbf{j}_f + \mathbf{j}_g, \qquad (f \odot_0 g)_i = \begin{cases} f_i & \text{if } 1 \leq i \leq \mathbf{j}_f \\ g_i & \text{if } \mathbf{j}_g < i \leq \mathbf{j}_g \end{cases}$$

These assignments underlie a functor $- \odot_0 - : \mathbf{pd}(n) \times \mathbf{pd}(n) \to \mathbf{pd}(n)$.

In order to justify calling the elements of $\mathbf{pd}(n)$ pasting diagrams, we will construct an equivalence between $\mathbf{pd}(n)$ and $\mathbf{pd}^{\mathrm{br}}(n)$. First, we note that, each of the preceding two examples corresponds to a natural operation on $\mathbf{pd}^{\mathrm{br}}(n)$ in a manner which we will shortly make precise (see Example 2.1.2.18). The concatenation operator $\odot_0$ on $\mathbf{pd}(n)$ corresponds to the operation $\odot_0$ on $\mathbf{pd}^{\mathrm{br}}$ which pastes along 0-boundaries. The functor Σ corresponds to the following *suspension construction*:

Definition 2.1.2.15 (See [30, §9.3])**.** For each bracketed n-pasting diagram $\pi \in \mathbf{pd}^{\mathrm{br}}(n)$, we define a bracketed $(n+1)$-pasting diagram $\Sigma\pi \in \mathbf{pd}^{\mathrm{br}}(n+1)$ inductively as follows:

- We set $\Sigma\mathbf{D}^n = \mathbf{D}^{n+1}$.
- We set $\Sigma(\pi^+) = (\Sigma\pi)^+$.
- We set $\Sigma(\pi \odot_k \pi') = (\Sigma\pi) \odot_{k+1} (\Sigma\pi')$.

For each π, the pasting diagram $\Sigma\pi$ has two distinct 0-cells $\star_0$ and $\star_1$. For each $n > 0$, an element x of $\Sigma\pi(n)$ corresponds exactly to an element $\bar{x}$ of $\pi(n-1)$. Thus, we define Σ on arrows by

$$(\Sigma f)(x) = f(\bar{x}), \qquad (\Sigma f)(\star_i) = \star_i.$$

These assignments underlie a fully faithful functor $\Sigma : \mathbf{pd}^{\mathrm{br}}(n) \to \mathbf{pd}^{\mathrm{br}}(n+1)$.

Example 2.1.2.16. Suppose that π is the following pasting diagram:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "g", ""{name=U, below}] \arrow[r, bend right, "h"', ""{name=D}] & C \arrow[Rightarrow, from=U, to=D, "\phi"] \end{tikzcd}$$

Then $\Sigma\pi$ is the following pasting diagram:

$$\begin{tikzcd}[column sep=huge]
\star_0 \arrow[r, bend left=60, "A", ""{name=A, below}] \arrow[r, "B" description, ""{name=B1, above}, ""{name=B2, below}] \arrow[r, bend right=60, "C"', ""{name=C, above}] & \star_1
\arrow[Rightarrow, from=A, to=B1, "f"]
\arrow[Rightarrow, from=B2, to=C, bend right=50, "g"', ""{name=g}]
\arrow[Rightarrow, from=B2, to=C, bend left=50, "h", ""{name=h}]
\arrow[Rrightarrow, from=g, to=h, "\phi"]
\end{tikzcd}$$

We now define a comparison functor $\Phi_n : \mathbf{pd}(n) \to \mathbf{pd}^{\mathrm{br}}(n)$ by induction on n. When $n = 0$, we define

$$\Phi_n(\star) = \mathbf{D}^0, \qquad \Phi_n(\mathrm{id}_\star) = \mathrm{id}_{\mathbf{D}^0}.$$

Suppose that $n > 0$. Then we define Φ_n on objects so that

$$\Phi_n(\pi_1, \ldots, \pi_l) = \Sigma(\Phi_{n-1}\pi_1) \odot_0 \cdots \odot_0 \Sigma(\Phi_{n-1}\pi_l). \tag{2.1.2.b}$$

Suppose that $f : \pi \to \rho$ is an arrow in $\mathbf{pd}'(n)$. For each i, let $\iota_i : \Sigma(\Phi_{n-1}\pi_i) \hookrightarrow \Phi_n\pi$ and $\kappa_i : \Sigma(\Phi_{n-1}\rho_{\mathbf{j}_f+i}) \hookrightarrow \Phi_n\rho$ be the canonical inclusions into the colimits defining $\Phi_n\pi$ and $\Phi_n\rho$ respectively. We define $\Phi_n f : \Phi_n\pi \to \Phi_n\rho$ to be the canonical map such that

$$(\Phi_n f) \circ \iota_i = \kappa_i \circ \Sigma(\Phi_{n-1} f_i).$$

Example 2.1.2.17. Suppose that π is the pasting diagram of Remark 2.1.2.8. Then $\Phi_n(\pi)$ is the bracketed pasting diagram of Example 2.1.2.5.

Example 2.1.2.18. It follows immediately from this definition that

$$\Phi_n(\pi \odot_0 \rho) = \Phi_n(\pi) \odot_0 \Phi_n(\rho),$$

and

$$\Phi_n(\Sigma\pi) = \Sigma\Phi_n(\pi).$$

The map Φ_n is easily seen to respect sources and targets. We define $\Phi : \mathbf{pd} \to \mathbf{pd}'$ to be the morphism of globular sets such that $\Phi(n) = \Phi_n$.

In order to prove that Φ_n is an equivalence, we will show that every pasting diagram in $\mathbf{pd}(n)$ can be written in a form similar to the right-hand side of (2.1.2.b).

Definition 2.1.2.19. Suppose that π is a (bracketed) n-pasting diagram. For each $-1 \leq k < n$, we say that π is *k-trivial* when there exists a (bracketed) $(n-k-1)$-pasting diagram $\bar{\pi} \in \mathbf{pd}(n-k-1)$ such that

$$\pi = \Sigma^{k+1}\bar{\pi}.$$

Note that $\bar{\pi}$ is necessarily unique.

It follows that a bracketed n-pasting diagram is k-trivial when it can built out of globes of dimension greater than k without pasting along i-boundaries for $i \leq k$.

Proposition 2.1.2.20. *Let π be a bracketed n-pasting diagram. Then there exists a unique n-pasting diagram $\Psi_n(\pi)$ such that*

$$\pi \cong \Phi_n(\Psi_n(\pi)).$$

Proof. We proceed by induction on n. The claim is clear when $n = 0$ since there is a unique 0-pasting diagram, namely $\star$, and there is a unique bracketed 0-pasting diagram, namely $\mathbf{D}^0$, and we have that

$$\Phi_0(\star) = \mathbf{D}^0.$$

Hence, suppose that $n > 0$. Then repeatedly applying associativity, unit and interchange laws, we have that

$$\pi \cong (\pi_1 \odot_0 (\pi_2 + \cdots \odot_0 \pi_l) \cdots)$$

where each $\pi_i = \Sigma\bar{\pi}_i$ is 0-trivial. Since each π_i is 0-trivial, there exists a unique $(n-1)$-pasting diagram $\Psi_n \bar{\pi}_i$ such that $\Phi_{n-1}(\Psi_{n-1}(\bar{\pi}_i)) \cong \bar{\pi}_i$. Thus, $\Phi_n(\Sigma\Psi_{n-1}(\bar{\pi}_i)) = \Sigma\Phi_{n-1}\Psi_{n-1}(\bar{\pi}_i) \cong \pi_i$. Hence, setting

$$\Psi_n(\pi) = (\Psi_{n-1}(\bar{\pi}_1), \dots, \Psi_{-1}(\bar{\pi}_l))$$

we have that $\Phi_n(\Psi_n(\pi)) \cong \pi$.

It remains to prove uniqueness. Hence, suppose that $\pi \cong \Phi_n(\rho)$ for some n-pasting diagram $\rho = (\rho_1, \dots, \rho_m)$. We will show that $l = m$, and $\rho_i = \Psi_{n-1}(\bar{\pi}_i)$. Each bracketed pasting diagram o induces a poset $\mathrm{Ord}_0\, o$ whose objects are k-cells of o for all $k \geq 0$, and whose partial order is generated by the relations

$$s_0 a \leq a \leq t_0 a,$$

for each $k > 0$ and each k-cell $a \in o(k)$. This assignment defines the objects part of a functor

$$\mathrm{Ord}_0 : \mathbf{pd}^{\mathrm{br}}(n) \longrightarrow \mathrm{Poset}\,.$$

Given $o \in \mathbf{pd}^{\mathrm{br}}(n)$, let $\mathbf{ht}\, o$ be the height of the poset $\mathrm{Ord}_0\, o$; that is, $\mathbf{ht}\, o$ is the maximum length of a chain in $\mathrm{Ord}_0(o)$. It is easily seen that $\mathbf{ht}\, \Phi_n(\Psi_n(\pi)) = 2l + 1$ and $\mathbf{ht}\, \Phi_n(\rho) = 2k + 1$. Since $\Phi_n(\Psi_n(\pi)) \cong \pi \cong \Phi_n(\rho)$ and the height of a poset is invariant under isomorphisms, we must have that $l = m$. For each bracketed n-pasting diagram o, and each cell $a \in o(k)$, let $\mathbf{ht}\, a$ be the height of a as an element of $\mathrm{Ord}_0\, o$; that is, $\mathbf{ht}\, a$ is one less than the maximum length of a chain in $\mathrm{Ord}_0(o)$ ending in a. Let $1 \leq i \leq l$. Let o_i be the subobject of o such that for each $k > 0$, we have that

$$a \in o_i(k) \iff \mathbf{ht}\, a = 2i - 1.$$

A 0-cell in o_i is the 0-source or 0-targets of one of these k-cells. Then, it follows that $\Phi_n(\Psi_n(\pi))_i \cong \pi_i$ and $\Phi_n(\rho)_i \cong \Sigma\Phi_{n-1}(\rho_i)$. However, since the heights of elements in a poset are invariant under isomorphism, we must have $\Phi_n(\rho)_i \cong \Phi_n(\Psi_n(\pi))_i$. We now have that

$$\Sigma\Phi_{n-1}(\rho_i) \cong \Phi_n(\rho)_i \cong \pi_i \cong \Sigma\Phi_{n-1}\Psi_{n-1}(\bar{\pi}_i).$$

Hence, since Σ is fully faithful, the uniqueness part of the inductive hypothesis implies that $\rho_i = \Psi_{n-1}\bar{\pi}_i$. Consequently, we have that $\rho = \Psi_n(\pi)$. ☐

Theorem 2.1.2.21. *The functor $\Phi_n : \mathbf{pd} \to \mathbf{pd}^{\mathrm{br}}(n)$ of (2.1.2.b) is an equivalence of categories.*

Proof. Proposition 2.1.2.20 implies that Φ_n is essentially surjective. Since, Φ_n is easily seen to be faithful, it remains to show that Φ_n is full. We will construct an arrow $\Psi_n(f) : \pi \to \rho$ such that $\Phi_n(\Psi_n(f)) = f$ by induction on n.

First, suppose that $n = 0$. Then, the only arrow in $\mathbf{pd}^{\mathrm{br}}(0)$ is $\mathrm{id}_{\mathbf{D}^0}$ and we have that $\Phi_n(\mathrm{id}_\star) = \mathrm{id}_{\mathbf{D}^0}$. Now suppose that $n > 0$. Suppose that $f : \Phi_n(\pi) \to \Phi_n(\rho)$ is a morphism in $\mathbf{pd}^{\mathrm{br}}(n)$ where $\pi = (\pi_1, \ldots, \pi_l)$ and $\rho = (\rho_1, \ldots, \rho_m)$. By definition of Φ_n, we have that

$$\Phi_n(\pi) = \Sigma\pi_1 \odot_0 \cdots \odot_0 \Sigma\pi_l, \qquad \Phi_n(\rho) = \Sigma\rho_1 \odot_0 \cdots \odot_0 \Sigma\rho_m. \tag{$\dagger$}$$

For each $0 \leq i \leq l$, let $\star_i$ be the unique 0-cell of $\Phi_n(\pi)$ such that $\mathbf{ht}\, \star_0 = 2i$. We define $\mathbf{j}_{\Psi f}$ to be the unique integer such that

$$\mathbf{ht}(f\star_0) = 2\mathbf{j}_{\Psi f}.$$

Suppose that $1 \leq i \leq l$. Then, there must exist a cell a in $\Phi_n\pi$ such that $s_0 a = \star_{i-1}$ and $t_0 a = \star_i$. Hence, fa is a cell in $\Phi_n\rho$ such that $s_0 fa = f\star_{i-1}$ and $t_0 fa = f\star_i$. Inspecting the 0-cells in ($\dagger$), it follows that $f\star_i = f\star_{i-1} + 2$. Hence, we have that $f\star_i = f\star_0 + 2i = 2(\mathbf{j}_{\Psi f} + i)$. Now, for any cell a in $\Sigma\pi_i$, we have that

$$2i - 2 = \mathbf{ht}\, \star_{i-1} \leq \mathbf{ht}\, a \leq \mathbf{ht}\, \star_i = 2i.$$

Consequently, since $\mathrm{Ord}_0 f$ is order preserving, we have that

$$2(\mathbf{j}_{\Psi f} + i) - 2 = \mathbf{ht}(f\star_{i-1}) \leq \mathbf{ht}(fa) \leq \mathbf{ht}(f\star_i) = 2(\mathbf{j}_{\Psi f} + i).$$

This implies that $fa \in \Sigma\rho_{\mathbf{j}_{\Psi f}+i}$. This allows us to define $(\Psi f)_i$ by restricting and co-restricting f. If $\iota_i : \Sigma\pi_i \hookrightarrow \pi$ and $\kappa_i : \Sigma\rho_{\mathbf{j}_{\Psi f}+i} \hookrightarrow \rho$ are the canonical inclusions, then we define $f_i : \Phi_{n-1}\pi_i \to \Phi_{n-1}\rho_{\mathbf{j}_{\Psi f}+i}$ to be the unique map such that

$$f \circ \iota_i = \kappa_i \circ \Sigma f_i.$$

We set $(\Psi_n f)_i = \Psi_{n-1} f_i$. Putting this together, we have that $\Psi_n f = (\Psi_{n-1} f_1, \dots, \Psi_{n-1} f_l)$. By construction, we have that $\Phi_n \Psi_n(f) = f$. □

Corollary 2.1.2.22. *For each $k \geq 0$, every $(k-1)$-trivial n-pasting diagram $\pi \in \mathbf{pd}(n)$ is of the form*

$$\pi = \pi_1 \odot_k \pi_2 + \cdots \odot_k \pi_l,$$

for some unique $l \geq 0$, and uniquely determined k-trivial $\pi_i \in \mathbf{pd}(n)$. Here, we use the convention that the 0-ary sum is the representable $\mathbf{D}^k$.

Proof. Suppose that π is k-trivial. When $k = -1$, this is Proposition 2.1.2.20. Otherwise, we have that $\pi = \Sigma\pi'$ for some $(k-1)$-trivial π'. The result now follows by induction. □

These results allow us to give a simple inductive description of the cells of each pasting diagram. First, note that since the unique 0-pasting diagram has a unique 0-cell, every pasting diagram π has a unique 0-cell in its 0-source pasting diagram $s_0\pi$. By abuse of notation, we refer to this 0-cell as $s_0\pi$. Similarly, every pasting diagram π has a unique target 0-cell, which we denote by $t_0\pi$. Now suppose that $\pi = (\pi_1, \dots, \pi_l)$ is any n-pasting diagram. Thus,

$$\pi = \Sigma\pi_1 \odot_0 \cdots \odot_0 \Sigma\pi_l.$$

Suppose that x is a 0-cell of π. If $x = s_0\pi$, then we write $x = (0)$. Note that $s_0\pi = s_0\Sigma\pi_1$ whenever $l > 0$. On the other hand, if $x \neq s_0\pi$, we must have that $x = t_0\Sigma\pi_i$ for some unique π_i. This follows from the fact that each π_i has exactly 2 different 0-cells, namely $s_0\pi_i$ and $t_0\pi_i$, and that for each $1 \leq i < l$, we have that $t_0\pi_i = s_0\pi_{i+1}$. In this case, we write $x = (i)$. Thus, there are exactly l distinct 0-cells in π, which we denote by $(0), (1), \dots, (l)$.

Now suppose that x is a k-cell of π for some $k > 0$. Then, we must have that $l > 0$ and, furthermore, $x \in \pi_i(k)$ for some unique i. Let π_i' be the unique pasting diagram such that $\pi_i = \Sigma\pi_i'$. Then x corresponds to some unique $(k-1)$-cell x' in π_i'. In this case, we write $x = (i, x')$.

Remark 2.1.2.23. From now on, we will not distinguish between $\mathbf{pd}$ and $\mathbf{pd}^{\mathrm{br}}$. We will identify each pasting diagram in $\mathbf{pd}$ with its corresponding globular set.

2.1.3 The free strict ω-category monad

Since pasting diagrams parameterise composition in higher categories, they allow us to give a simple description of the free strict ω-category monad $\mathbf{T} : \mathbb{G}\text{-Set} \to \mathbb{G}\text{-Set}$. For any globular set X, an n-cell f of $\mathbf{T}X$ consists of an n-pasting diagram $\pi \in \mathbf{pd}(n)$, called the *shape* of f, together with a map

$$f : \pi \longrightarrow X.$$

We refer to these maps as *pasting diagrams in* X. For each cell $i : \mathbf{D}^l \to \pi$ of π, we denote by f_i the composite

$$l \xrightarrow{i} \pi \xrightarrow{f} X.$$

We will also write $f = (f_i)_{i \in \pi}$ in order to emphasise that f amounts to a collection of cells of X indexed by the cells of π. When $n > 0$, the source and target of f in $\mathbf{T}X$ are the following $\pi_{\partial_{n-1}}$-shaped $(n-1)$-cells:

$$sf = f \circ \sigma_{\pi,n-1}, \qquad tf = f \circ \tau_{\pi,n-1}.$$

In summary, each level of the functor $\mathbf{T}$ is a coproduct of representables:

$$\mathbf{T}(-)(n) = \coprod_{\pi \in \mathbf{pd}(n)} \mathbb{G}\text{-Set}(\pi, -)$$

Thus, $\mathbf{T}$ is *familially representable* in the sense of [12, 30]. In particular, when $\top$ is the terminal globular set, we have that $\mathbf{T}\top = \mathbf{pd}$.

Suppose that $f, g \in \mathbf{T}X(n)$ are n-pasting diagrams in X such that $t_k f = s_k g$. Then we have corresponding pasting diagrams

$$f : \pi_1 \longrightarrow X, \qquad g : \pi_2 \longrightarrow X,$$

such that $f \circ \tau_{\pi_1,k} = g \circ \sigma_{\pi_2,k}$. We define $f \odot_k g$ to be the induced map

$$\pi_1 \odot_k \pi_2 \longrightarrow X.$$

Then, $f \odot_k g$ defines another element of $\mathbf{T}X(n)$. More generally, suppose that we have a pasting diagram $\Gamma : \rho \to \mathbf{T}X$ sending each $i \in \rho$ to a π_i-shaped diagram Γ_i in X. (When $\rho = \mathbf{D}^n \odot_k \mathbf{D}^n$, and Γ sends the n-cell of the first component to π_1 and sends the n-cell of the second component to π_2, we recover the preceding example.) Then we have a corresponding pasting diagram of pasting diagrams, $\mathbf{T}! \circ \rho : \rho \to \mathbf{T}\top = \mathbf{pd}$, sending each $i \in \rho$ to the pasting diagram π_i. Hence, taking the colimit of the composite

$$\mathrm{el}(\rho) \xrightarrow{\mathrm{el}(\Gamma)} \mathrm{el}(\mathbf{T}X) \xrightarrow{\mathrm{el}(\mathbf{T}!)} \mathrm{el}(\mathbf{T}!) \xrightarrow{\mathcal{I}} \mathbb{G}\text{-Set},$$

we obtain a globular set π. Since this colimit can be decomposed into pushouts of the appropriate form (see Definition 2.1.2.1 above and [56]), we have that π is a pasting diagram. Let $\iota_i : \pi_i \to \pi$ be the canonical coprojection from π_i into the colimit π. Suppose that j is a cell in π. Then, there exist cells i in ρ and h in π_i such that $\iota_i(h) = j$. Hence, we have a cell $\Gamma_i(h)$ in $\mathbf{T}X$. The conditions on the colimit π ensure that this cell of $\mathbf{T}X$ does not depend on the choice of i. These assignments can be assembled into a diagram $\pi \to \mathbb{X}$, which we denote by:

$$\bigodot_{i\in\rho} \Gamma_i.$$

The multiplication of the monad $\mathbf{T}$ sends Γ to $\bigodot_{i\in\rho} \Gamma_i$. Thus, we think of the multiplication of $\mathbf{T}$ as a *globular sum*. In fact, these colimits are precisely the globular sums used in the various definitions of ∞-category based on Grothendieck's work and developed by Maltsiniotis and Ara, amongst others. (See [1, 36].)

Suppose that $n \leq m$. Then, for each n-cell M in X, there is a canonical $\mathbf{D}^n$-shaped m-pasting diagram in X which maps the unique n-cell of $\mathbf{D}^n$ to M. We denote this m-pasting diagram by

$$[M] \in \mathbf{T}(X)(m).$$

We will often denote this pasting diagram simply by M when there is no ambiguity. The unit of the monad $\mathbf{T}$ is the assignment $X \to \mathbf{T}X$ which sends each n-cell M to the n-pasting diagram $[M]$. The unit laws say that, for each π-shaped pasting diagram $\Gamma : \pi \to X$, we have that

$$\bigodot_{i\in\pi} [\Gamma_i] = \Gamma = \bigodot_{j\in\mathbf{D}^n} [\Gamma]_j.$$

For a more detailed description of the free strict ω-category monad, see [56].

Remark 2.1.3.1. We can give a similar description of the free strict n-category monad $\mathbf{T}_n$ on n-globular sets, by considering $\mathbf{pd}_n$ instead of $\mathbf{pd}$. For any $k \leq n$, the truncation functor $\mathrm{tr}_k : \mathbb{G}_n\text{- Set} \to \mathbb{G}_k\text{- Set}$ induces a morphism of monads $\mathbf{T}_n \to \mathbf{T}_k$.

2.2 Definition

A crucial property of the free strict ω-category monad, $\mathbf{T}$, is that it is *cartesian*; i.e., its underlying functor preserves pullbacks and the naturality squares of its unit and multiplication are pullback squares (see [30]). This allows us to define a notion of generalised multicategory using Leinster's theory of $\mathbf{T}$-multicategories (see [30]).

Definition 2.2.0.1. A $\mathbf{T}$-*span* is a span of n-globular sets of the following form:

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
\mathbf{T}A & & B
\end{tikzcd}$$

We can compose $\mathbf{T}$-spans,

$$\begin{tikzcd}
 & X \arrow[dl, "a"'] \arrow[dr, "b"] & & & Y \arrow[dl, "b'"'] \arrow[dr, "c"] & \\
\mathbf{T}A & & B & \mathbf{T}B & & C
\end{tikzcd}$$

by computing a pullback as in the following diagram

$$\begin{tikzcd}
 & & \mathbf{T}X \times_{\mathbf{T}B} Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & \mathbf{T}X \arrow[dl, "\mathbf{T}a"'] \arrow[dr, "\mathbf{T}b"] & & Y \arrow[dl, "b'"'] \arrow[dr, "c"] & \\
\mathbf{T}^2A \arrow[d, "\mu_A"'] & & \mathbf{T}B & & C \\
\mathbf{T}A & & & &
\end{tikzcd}$$

and then composing the left leg with the multiplication μ of $\mathbf{T}$. Let η_X be the unit of $\mathbf{T}$ at X. Then the identity $\mathbf{T}$-span at X is the following diagram:

$$\begin{tikzcd}
 & X \arrow[dl, "\eta_X"'] \arrow[dr, "\mathrm{id}_X"] & \\
\mathbf{T}X & & X
\end{tikzcd}$$

Putting all these data together, we obtain a bicategory $\mathbf{T}\text{-}\mathrm{Span}$.

We can use this bicategory to give our main definition succinctly.

Definition 2.2.0.2. An *globular multicategory* is a monad in the bicategory $\mathbf{T}\text{-}\mathrm{Span}$.

Let us now unpack this definition. Each globular multicategory $\mathbb{X}$ has an underlying $\mathbf{T}$-span

$$\begin{tikzcd}
 & X_1 \arrow[dl, "\mathrm{Ctx}"'] \arrow[dr, "\mathrm{Ty}"] & \\
\mathbf{T}X_0 & & X_0
\end{tikzcd}$$

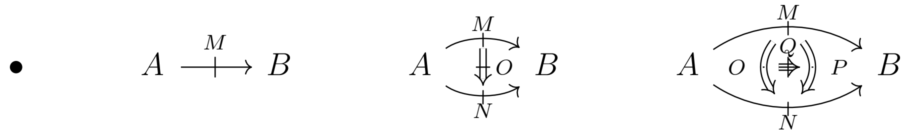
for some pair of globular sets X_0, X_1. We refer to $\mathbf{T}$-spans of this form as *globular multigraphs*. We will use type-theoretic terminology to refer to the data contained in globular multigraphs.

2.2.1 Types

We define $\mathrm{Type}\,\mathbb{X} = X_0$. A *$k$-type* is an element of $X_0(k)$. When M is a k-type such that $sM = A$ and $tM = B$, we denote M by a stroked arrow:

$$M : A \nrightarrow B.$$

In low dimensions, we also depict k-types using the notation for globes described in Section 2.1, except that we use stroked arrows. Hence, going from left to right, the figure below depicts an unlabelled 0-type, a 1-type $M : A \to B$, a 2-type $O : M \to N$ where $M, N : A \nrightarrow B$ are 1-types, and a 3-type $Q : O \nrightarrow P$ where $O, P : M \nrightarrow N$ are 2-types:



A π-shaped *k-context* $\Gamma = (\Gamma_i)_{i\in\pi}$ is a π-shaped element of $\mathbf{T}X_0(k)$. That is a map of the form

$$\Gamma : \pi \longrightarrow X_0.$$

We depict contexts as pasting diagrams of types.

Example 2.2.1.1. When $\pi = \mathbf{D}^1 \odot_0 \mathbf{D}^2$, and $\Gamma : \pi \to X_0$ is the π-shaped 2-context defined by

$$\Gamma(0) = A, \qquad \Gamma(1) = B, \qquad \Gamma(2) = C,$$
$$\Gamma(1,0) = M, \qquad \Gamma(2,0) = N, \qquad \Gamma(2,1) = N,$$
$$\Gamma(2,1,0) = O,$$

we depict Γ as follows:

$$\begin{tikzcd}
A \arrow[r, "M", "+"{marking}] & B \arrow[r, bend left, "N", "+"{marking}, ""{name=U, below}] \arrow[r, bend right, "N"', "+"{marking}, ""{name=D}] & C. \arrow[from=U, to=D, Rightarrow, "O", "+"{marking}]
\end{tikzcd} \tag{2.2.1.a}$$

When an n-context Γ contains a k-type D such that $k < n$ and there does not exist a type E such that $sE = D$ or $te = D$, we sometimes depict Γ with two copies of the type D joined by an $=$ sign. Thus, the context Eq. (2.2.1.a) could also be depicted as:

$$\begin{tikzcd}
A \arrow[r, bend left, "M", "+"{marking}, ""{name=U1, below}] \arrow[r, bend right, "M"', "+"{marking}, ""{name=D1}] & B \arrow[r, bend left, "N", "+"{marking}, ""{name=U, below}] \arrow[r, bend right, "N"', "+"{marking}, ""{name=D}] & C. \arrow[from=U1, to=D1, Rightarrow, no head, equal] \arrow[from=U, to=D, Rightarrow, "O", "+"{marking}]
\end{tikzcd} \tag{2.2.1.b}$$

A *k-variable* in a π-shaped context Γ is a k-cell in π. When x a k-variable and $A = \Gamma_x$, we say that A is the *type of* x, and write

$$x : A.$$

For example, in Eq. (2.2.1.a), we have that $(0) : A$ and $(2, 1, 0) : O$. We can use variables to distinguish between different copies of the same type in a context. For example, in Eq. (2.2.1.a), there are two variables with type N, namely $(2, 0)$ and $(2, 1)$.

2.2.2 Terms

We define $\operatorname{Term}\mathbb{X} = X_1$. A *$k$-term* f is an element of $X_1(k)$. Each term has a context $\operatorname{Ctx} f$ and an output type $\operatorname{Ty} f$. We say that f is *π-shaped* when $\operatorname{Ctx} f$ is π-shaped. When $\operatorname{Ctx} f = \Gamma$ and $\operatorname{Ty} f = A$, we write

$$f : \Gamma \longrightarrow A.$$

Thus, we think of f as a generalised arrow sending a π-shaped k-context Γ (a pasting diagrams of typed input variables) to an output k-type A. Since $\operatorname{Term}\mathbb{X}$ is a globular set, terms also have source and target terms: for each $k > 0$ and each k-term $f : \Gamma \to A$, we have $(k-1)$-terms $sf : s\Gamma \to sA$ and $tf : t\Gamma \to tA$. In this case we write

$$f : sf \nrightarrow tf.$$

Source and target terms satisfy globularity conditions. We depict terms as vertical arrows between contexts.

Example 2.2.2.1. The following diagram depicts a $(\mathbf{D}^1 \odot_0 \mathbf{D}^1)$-shaped 1-term $f : M \odot_0 N \to O,\ g \;\;\nrightarrow h$ where $g : A \to D$ and $h : C \to E$:

$$\begin{tikzcd}
A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "g"'] & B \arrow[r, "N", "\shortmid"{marking}] \arrow[d, Rightarrow, "f"] & C \arrow[d, "h"] \\
D \arrow[rr, "P"', "\shortmid"{marking}] & {} & E
\end{tikzcd}$$

Example 2.2.2.2. The following diagram depicts a $(\mathbf{D}^1 \odot_0 \mathbf{D}^2)$-shaped 2-term $f : M \odot_0 O \to R,\ sf \;\;\nrightarrow tf$ where $sf : M \odot_0 N \to Q,\ s^2 f \;\;\nrightarrow t^2 f$ and $tf : M \odot_0 N \to P,\ s^2 f \;\;\nrightarrow t^2 f$:

$$\begin{tikzcd}
A \arrow[dr, "M", "\shortmid"{marking}] \arrow[dd, "s^2 f"{description}] & & \\
& B \arrow[dr, bend left=40, "N", "\shortmid"{marking}, ""{name=N1, below}] \arrow[dr, bend right=20, "N"', "\shortmid"{marking}, ""{name=N2, above}] & \\
D \arrow[drr, bend left=15, "Q", "\shortmid"{marking}, ""{name=Q, below}] \arrow[drr, bend right=25, "P"', "\shortmid"{marking}, ""{name=P, above}] & & C \arrow[d, "t^2 f"{description}] \\
& & E
\arrow[Rightarrow, from=N1, to=N2, "O", "\shortmid"{marking}]
\arrow[Rightarrow, from=2-2, to=Q, "sf"', ""{name=SF}]
\arrow[Rightarrow, bend right, from=2-2, to=P, "tf"', ""{name=TF}]
\arrow[Rrightarrow, from=SF, to=TF, "f"]
\arrow[Rightarrow, from=Q, to=P, "R", "\shortmid"{marking}]
\end{tikzcd}$$

Whenever, $g : s\Gamma \to sA$ and $h : t\Gamma \to tA$ are terms satisfying $sg = sh$ and $tg = th$, we say that g and h are *term-wise parallel.* In particular, source and target terms are term-wise parallel: when $k > 1$, we have that $s^2 f = stf$ and $t^2 f = tsf$. For any term-wise parallel g, h, we denote the set of terms f such that $f : \Gamma \to A$ and $f : g \;\;\nrightarrow h$ by

$$[\Gamma \longrightarrow A, \quad g \longrightarrow\!\!\!\!\!\!\!\!\not\;\;\;\; h].$$

For any pasting diagram $\pi \in \mathbf{pd}(k)$, a π-shaped *substitution* is a π-shaped element $f = (f_i)_{i \in \pi} = \bigodot_{i \in \pi} f_i$ of $\mathbf{T}X_1(n)$. For each $i \in \mathrm{el}(\pi)$, we have that

$$f_i : \Gamma_i \longrightarrow \Delta_i.$$

By pasting together the domain contexts of these terms, we obtain a context

$$\Gamma = \bigodot_{i \in \pi} \Gamma_i.$$

By pasting together the codomain types, we obtain a π-shaped context

$$\Delta = \bigodot_{i \in \pi} \Delta_i.$$

Whenever Γ and Δ are defined in this way, we write

$$f : \Gamma \longrightarrow \Delta.$$

Since variables in a π-shaped context are elements of π, we will frequently index by contexts rather than pasting diagrams. For example, in this case, we have that

$$\Gamma = \bigodot_{i \in \Delta} \Gamma_i, \qquad \Delta = \bigodot_{i \in \Delta} \Delta_i, \qquad f = \bigodot_{i \in \Delta} f_i.$$

For all $n > 0$, each n-substitution $f : \Gamma \to \Delta$ has source and target $(n-1)$-substitutions $sf : s\Gamma \to s\Delta$ and $tf : t\Gamma \to t\Delta$. As with terms, we write

$$f : sf \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} tf.$$

Source and target substitutions satisfy globularity conditions.

Example 2.2.2.3. Suppose that we have 1-terms $\phi : M \odot_0 N \to P$, $f \mathrel{\mapstochar\rightarrow} g$ and $\psi : O \to Q$, $g \mathrel{\mapstochar\rightarrow} h$. Then there is a $(\mathbf{D}^1 \odot_0 \mathbf{D}^1)$-shaped 1-substitution $\phi \odot_0 \psi : M \odot_0 N \odot_0 O \to P \odot_0 Q$, $f \mathrel{\mapstochar\rightarrow} h$; we depict this substitution as follows:

$$\begin{tikzcd}
A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[r, "N", "\shortmid"{marking}] \arrow[d, Rightarrow, "\phi"] & C \arrow[rr, "O", "\shortmid"{marking}] \arrow[d, "g"'] & & D \arrow[d, "h"] \\
E \arrow[rr, "P"', "\shortmid"{marking}] & {} & F \arrow[rr, "Q"', "\shortmid"{marking}] & {} & G
\arrow[from=1-4, to=2-4, Rightarrow, "\psi"]
\end{tikzcd}$$

2.2.3 Composition of terms

So far we have described the data contained in globular multigraphs, but we now consider the distinctive feature of globular multicategories: they admit a notion of *composition of terms*. Suppose that we have a substitution $f : \Gamma \to \Delta$ and a term $g : \Delta \to A$ in a globular multicategory $\mathbb{X}$. The multiplication of $\mathbb{X}$, qua monad in $\mathbf{T}$-Span, allows us to define a *composite* term

$$f;g : \Gamma \longrightarrow \Delta, \qquad sf;sg \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} tf;tg$$

We think of $f;g$ as the result of substituting the term f_i for the variable i in (the domain context of) g. We depict composite terms by vertical concatenation.

Example 2.2.3.1. Suppose that $f : A \to B$ is a 0-substitution; that is a 0-term. Suppose that $g : B \to C$ is a 0-term. Then we depict the composite $f;g$ by

$$\begin{tikzcd}
A \arrow[d, "f"] \\
B \arrow[d, "g"] \\
C
\end{tikzcd}$$

Example 2.2.3.2. Suppose that $\phi \odot_0 \psi : M \odot_0 N \odot_0 O \to P \odot_0 Q, f \nrightarrow g$ is the 1-substitution depicted in Example 2.2.2.3. Suppose that we have a 1-term $\xi : P \odot_0 Q \to R, d \nrightarrow e$. Then we depict the composite $(\phi \odot_0 \psi); \xi$ as follows:

$$\begin{tikzcd}
A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[d, Rightarrow, "\phi"] \arrow[r, "N", "\shortmid"{marking}] & C \arrow[rr, "O", "\shortmid"{marking}] \arrow[d, "g"'] & {} \arrow[d, Rightarrow, "\psi"] & D \arrow[d, "h"] \\
E \arrow[rr, "P"', "\shortmid"{marking}] \arrow[d, "d"'] & {} \arrow[d, Rightarrow, "\xi"] & F \arrow[rr, "Q"', "\shortmid"{marking}] & {} & G \arrow[d, "e"] \\
H \arrow[rrrr, "R"', "\shortmid"{marking}] & {} & & & I
\end{tikzcd}$$

We can also compose pairs of substitutions. Suppose that $f : \Gamma \to \Delta$ and $g : \Delta \to E$ are n-substitutions. Then, we define

$$f; g : \Gamma \longrightarrow E$$

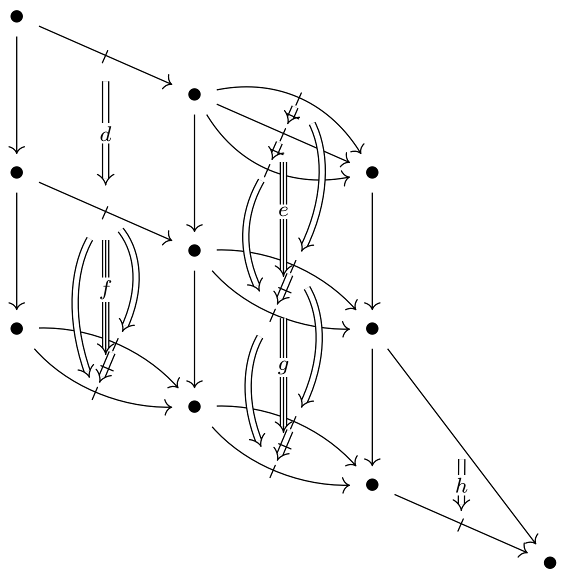
by

$$f; g = \bigodot_{i \in E} (f; g)_i$$

where, for each variable $i \in E$, we have that

$$(f; g)_i = (f_{j_i})_{j_i \in \Delta_i}; g_i.$$

Example 2.2.3.3. Consider the following diagram:



The shape of the 1-term d is $\mathbf{D}^1$. The shape of the 2-term e is $\mathbf{D}^2 \odot_1 \mathbf{D}^2$. The top row of the picture denotes the $(\mathbf{D}^1 \odot_0 \mathbf{D}^2)$-shaped substitution $d \odot_0 e$. The shape of the

2-term f is $\mathbf{D}^1$. The shape of the 2-term g is $\mathbf{D}^2$. The shape of the 1-term h is $\mathbf{D}^0$. The bottom row of the picture denotes the $(\mathbf{D}^2 \odot_0 \mathbf{D}^2 \odot_0 \mathbf{D}^1)$-shaped 2-substitution $f \odot_0 g \odot_0 h$. The whole picture denotes the composite $(d \odot_0 e); (f \odot_0 g \odot_0 h)$. By definition, we have that

$$(d \odot_0 e); (f \odot_0 g \odot_0 h) = (d; f) \odot_0 (e; g) \odot_0 (t_0 e; h).$$

Remark 2.2.3.4. As this last example illustrates, it follows from the definition of $-;-$ that there is an *interchange law between* $-;-$ *and* $\bigodot$. Suppose that $f : \Gamma \to E$ and $g : E \to \Delta$ are composable n-substitutions. For each $x \in \Delta$, let $f_x = \bigodot_{y \in E_x} f_y$. Then

$$f = \bigodot_{y \in E} f_y = \bigodot_{x \in \Delta} \bigodot_{y \in E_x} f_y \; = \bigodot_{x \in \Delta} f_x.$$

Hence,

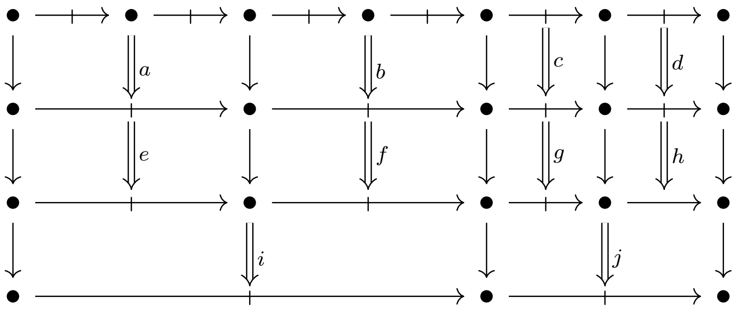
$$\begin{aligned} \bigodot_{x \in \Delta} f_x; g_x &= \bigodot_{x \in \Delta} \left(\bigodot_{y \in E_x} (f_x)_y \right); g_x \\ &= f; g \\ &= \left(\bigodot_{x \in \Delta} f_x \right); \left(\bigodot_{x \in \Delta} g_x \right). \end{aligned}$$

The *associativity law* of globular multicategories says that for all $f : \Gamma \to \Delta$, $g : \Delta \to E$ and $h : E \to A$, we have that

$$(f; g); h = f; (g; h).$$

This identity holds both when h is a term and when h is a substitution. We will tend to omit these brackets when working with $-;-$.

Example 2.2.3.5. Consider the following diagram:



The whole diagram depicts a composite $(\mathbf{D}^1 \odot_0 \mathbf{D}^1)$-shaped 1-substitution. Associativity implies that any two ways of building up this composite from its parts are the same.

For each n-type A, the unit of the globular multicategory $\mathbb{X}$, qua monad, induces us an *identity n-term*

$$\mathrm{id}_A : [A] \longrightarrow A, \qquad \mathrm{id}_{sA} \twoheadrightarrow \mathrm{id}_{tA}.$$

For each n-context Γ, we define the *identity n-substitution*

$$\mathrm{id}_\Gamma : \Gamma \longrightarrow \Gamma, \qquad \mathrm{id}_{s\Gamma} \twoheadrightarrow \mathrm{id}_{t\Gamma}$$

by setting $(\mathrm{id}_\Gamma)_i = \mathrm{id}_{\Gamma_i}$, for each $i \in \Gamma$. The unit laws of $\mathbb{X}$ says that for any n-term $f : \Gamma \to A$,

$$f; \mathrm{id}_A = f = \mathrm{id}_\Gamma; f.$$

Similar equations also hold when f is a substitution.

Example 2.2.3.6. Suppose that ϕ is the following 1-term:

$$\begin{tikzcd}
A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[r, "N", "\shortmid"{marking}] \arrow[d, Rightarrow, "\phi"'] & C \arrow[d, "g"] \\
D \arrow[rr, "O"', "\shortmid"{marking}] & {} & E
\end{tikzcd}$$

Then the unit laws say that

$$\begin{tikzcd}
A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[r, "N", "\shortmid"{marking}] \arrow[d, Rightarrow, "\phi"'] & C \arrow[d, "g"] \\
D \arrow[rr, "O"', "\shortmid"{marking}] \arrow[d, "\mathrm{id}_D"'] & {} \arrow[d, Rightarrow, "\mathrm{id}_O"'] & E \arrow[d, "\mathrm{id}_E"] \\
D \arrow[rr, "O"', "\shortmid"{marking}] & {} & E
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[r, "N", "\shortmid"{marking}] \arrow[d, Rightarrow, "\phi"'] & C \arrow[d, "g"] \\
D \arrow[rr, "O"', "\shortmid"{marking}] & {} & E
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, "M", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & {} \arrow[d, Rightarrow, "\mathrm{id}_M"'] & B \arrow[rr, "N", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_B", "\shortmid"{marking}] & {} \arrow[d, Rightarrow, "\mathrm{id}_N"] & C \arrow[d, "\mathrm{id}_C"] \\
A \arrow[rr, "M", "\shortmid"{marking}] \arrow[d, "f"'] & {} & B \arrow[rr, "N", "\shortmid"{marking}] \arrow[d, Rightarrow, "\phi"'] & {} & C \arrow[d, "g"] \\
D \arrow[rrrr, "O"', "\shortmid"{marking}] & & {} & & E
\end{tikzcd}$$

Remark 2.2.3.7. We sometimes depict the arrows of identity terms using vertical $=$ signs. For example, when $M : A \to B$ is a 1-type, the following diagram represents id_M:

$$\begin{tikzcd}
A \arrow[rr, "M", "\shortmid"{marking}] \arrow[d, equal] & {} \arrow[d, equal] & B \arrow[d, equal] \\
A \arrow[rr, "M"', "\shortmid"{marking}] & {} & B
\end{tikzcd}$$

As another example, given $M : A \to B$ as above, a 1-term $f : A \to M$, $\mathrm{id}_A \twoheadrightarrow g$ could be depicted as follows:

$$\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, equal] & {} \arrow[d, Rightarrow, "f"'] & A \arrow[d, "g"] \\
A \arrow[rr, "M"', "\shortmid"{marking}] & {} & B
\end{tikzcd}$$

Remark 2.2.3.8. Given a globular multicategory, $\mathbb{X}$, the globular set $\operatorname{Type}\mathbb{X}$ can be made into a globular category by defining an arrow $A \to B$ in $\operatorname{Type}\mathbb{X}(n)$ to be an n-term $[A] \to B$ in $\mathbb{X}$.

Remark 2.2.3.9. For consistency of notation, we say that a globular multicategory $\mathbb{X}$ has a unique (-1)-type (or context, term, or substitution) which we denote by $\star$. Every 0-type (or context, term, or substitution) A satisfies $A : \star \nrightarrow \star$.

Definition 2.2.3.10. Given globular multigraphs $\mathbb{X}, \mathbb{Y}$, a *map* of globular multigraphs, $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$, is a pair of arrows, F_0, F_1, making the following diagram commute:

$$\begin{tikzcd}
 & X_1 \arrow[dl, "\mathrm{Ctx}"'] \arrow[dr, "\mathrm{Ty}"] \arrow[dd, "F_1"] & \\
\mathbf{T}X_0 \arrow[dd, "\mathbf{T}F_0"'] & & X_0 \arrow[dd, "F_0"] \\
 & Y_1 \arrow[dl, "\mathrm{Ctx}"] \arrow[dr, "\mathrm{Ty}"'] & \\
\mathbf{T}Y_0 & & Y_0
\end{tikzcd}$$

A *homomorphism* of globular multicategories is a map of globular multigraphs preserving composition and identities of terms in $\mathbb{X}$; equivalently, homomorphisms preserve the multiplication and unit of $\mathbb{X}$, qua monad.

We denote the category of globular multicategories and homomorphisms by GlobMult. Most constructions of this thesis can be understood using ordinary 1-category theory. However, certain constructions are better understood by considering GlobMult to be a strict 2-category. The following definition describes the 2-cells of this 2-category:

Definition 2.2.3.11. Let $F, G : \mathbb{X} \to \mathbb{Y}$ be homomorphisms of n-globular multicategories. Then, a *transformation* $\phi : F \Rightarrow G$ consists of the following data:

- For each k-type A in $\mathbb{X}$, we require a k-term

  $$\phi_A : FA \longrightarrow GA, \qquad \phi_{sA} \nrightarrow \phi_{tA}.$$

  It follows that, for each context Γ in $\mathbb{X}$, there is an induced substitution

  $$\phi_\Gamma : F\Gamma \longrightarrow G\Gamma, \qquad \phi_{s\Gamma} \nrightarrow \phi_{t\Gamma}$$

  defined by $(\phi_\Gamma)_x = \phi_{\Gamma_x}$.

- We require the following naturality condition: whenever $f : \Gamma \to A$ is a term in X, we have that
$$Ff; \phi_A = \phi_\Gamma; Gf.$$

Remark 2.2.3.12. We can recover the whole 2-category GlobMult using the theory of generalised multicategories developed in [17].

Remark 2.2.3.13. Replacing the free strict ω-category monad $\mathbf{T}$ in these definitions with the free strict n-category monad $\mathbf{T}_n$, we obtain notions of *n-globular multigraph* and *n-globular multicategory*, for each finite n. For consistency, we will sometimes refer to plain globular multicategories as *ω-globular multicategories*. We denote the (2-)category of n-globular multicategories by $n\text{ - GlobMult}$. The truncation functors of Remark 2.1.1.3 and Remark 2.1.3.1 induce truncation functors

$$0\text{ - GlobMult} \xleftarrow{\mathrm{tr}_0} 1\text{ - GlobMult} \xleftarrow{\mathrm{tr}_1} \cdots \longleftarrow \omega\text{ - GlobMult}\,.$$

These functors have fully faithful left adjoints L_{tr_k} such that

$$\mathrm{Type}(L_{\mathrm{tr}_k}\mathbb{X})(i) = \begin{cases} (\mathrm{Type}\,\mathbb{X})(i) & \text{if } i \leq k \\ \emptyset & \text{if } i > k \end{cases}$$
$$\mathrm{Term}(L_{\mathrm{tr}_k}\mathbb{X})(i) = \begin{cases} (\mathrm{Term}\,\mathbb{X})(i) & \text{if } i \leq k \\ \emptyset & \text{if } i > k \end{cases}$$

We define the *dimension* of a globular multicategory by

$$\dim \mathbb{X} = \dim \mathrm{Type}\,\mathbb{X}.$$

It follows that $\dim \mathbb{X} = n$ if and only if there is an n-globular multicategory $\mathbb{X}'$ such that $L_{\mathrm{tr}_n}\mathbb{X}' = \mathbb{X}$. We can typically use this observation to obtain results about n-globular multicategories from results about ω-globular multicategories. Hence, as with globular sets, we focus on the infinite-dimensional case.

2.3 First Examples

Example 2.3.0.1. A 0-globular multicategory is just a category.

Example 2.3.0.2. A 1-globular multicategory is a *virtual double category.* A virtual double category consists of the following data:

- A category whose objects we call 0-*types*, and whose arrows we call 0-*terms*. We depict 0-terms as vertical arrows:

$$\begin{tikzcd}
A \arrow[d] \\
B
\end{tikzcd}$$

- A collection of arrows whose sources and targets are 0-types, which we will refer to as 1-*types*. We depict 1-types as barred horizontal arrows such as

$$\begin{tikzcd}
A \arrow[r, "M", "|"{marking}] & B
\end{tikzcd}$$

- A collection of arrows, called 1-terms, sending composable lists of 1-types to a 1-type. Each 1-term also has a source and target 0-term. A typical 1-term could be depicted as follows:

$$\begin{tikzcd}
A \arrow[r, "M", "|"{marking}] \arrow[d, "f"'] & B \arrow[r, "N", "|"{marking}] \arrow[d, phantom, "\Downarrow"] & C \arrow[d, "g"] \\
D \arrow[rr, "O"', "|"{marking}] & & E
\end{tikzcd}$$

  A 1-term whose source list has length 0 could be depicted as follows:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "g"] \\
C \arrow[r, "M"', "|"{marking}] & D
\end{tikzcd}$$

- The 1-terms can be composed vertically; A typical composite could be depicted as follows:

$$\begin{tikzcd}
\bullet \arrow[r, "|"{marking}] \arrow[d] & \bullet \arrow[r, "|"{marking}] \arrow[d, phantom, "\Downarrow"] & \bullet \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[r, "|"{marking}] \arrow[d] & \bullet \arrow[r, "|"{marking}] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[r, "|"{marking}] & \bullet \arrow[d] \\
\bullet \arrow[rr, "|"{marking}] \arrow[d] & & \bullet \arrow[r, "|"{marking}] & \bullet \arrow[rrr, "|"{marking}] \arrow[d, phantom, "\Downarrow"] & & & \bullet \arrow[d] \\
\bullet \arrow[rrrrrr, "|"{marking}] & & & & & & \bullet
\end{tikzcd}$$

  This composition is associative, and there are identity 1-terms of the following form:

$$\begin{tikzcd}
A \arrow[r, "M", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathrm{id}_M"] & B \arrow[d, equal] \\
A \arrow[r, "M"', "|"{marking}] & B
\end{tikzcd}$$

This one-dimensional case is thoroughly studied in [17].

Example 2.3.0.3. Every pseudo-double category (with strict vertical composition and weak horizontal composition) can be viewed as a virtual double category.

Example 2.3.0.4. Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are virtual double categories underlying pseudo-double categories. Then a homomorphism $\mathbb{X} \to \mathbb{Y}$ is a lax functor between the corresponding double categories.

Example 2.3.0.5. Let A and B be parallel n-types in $\mathbb{X}$, for some $n < \dim \mathbb{X}$. Then there is a canonical $(\dim \mathbb{X} - n)$-globular multicategory $\mathbb{X}(A, B)$ such that:

- A 0-type in $\mathbb{X}(A, B)$ is an $(n+1)$-type $M : A \nrightarrow B$ in $\mathbb{X}$.
- A 0-term in $\mathbb{X}(A, B)$ is an $(n+1)$-term $f : [M] \to [N], \mathrm{id}_A \nrightarrow \mathrm{id}_B$ in $\mathbb{X}$.
- When $0 < k \leq \dim \mathbb{X} - n$, a k-type M of $\mathbb{X}(A, B)$ is an $(n+k+1)$-type of $\mathbb{X}$ such that $s_n M = A$ and $t_n M = B$.
- When $0 < k \leq \dim \mathbb{X} - n$, a π-shaped k-term f is a $\Sigma^{n+1}\pi$-shaped $(n+k+1)$-term in $\mathbb{X}$ such that $s_n f = \mathrm{id}_A$ and $t_n f = \mathrm{id}_B$.

The following example, which is essentially contained in [54], motivates our use of type-theoretic terminology:

Example 2.3.0.6. Every dependent type theory $\mathcal{T}$ induces a globular multicategory $\mathcal{G}_{\mathrm{ML}}(\mathcal{T})$. We have that:

- A 0-type A in $\mathcal{G}_{\mathrm{ML}}(\mathcal{T})$ is a type
$$\vdash A^\circ : \mathrm{Type}$$
in $\mathcal{T}$.
- For $n > 0$, an $(n+1)$-type $M : A \nrightarrow B$ in $\mathcal{G}_{\mathrm{ML}}(\mathcal{T})$ is a dependent type judgement
$$x : A^\circ, y : B^\circ \vdash M^\circ(x, y) : \mathrm{Type}$$
in $\mathcal{T}$.
- Each globular context $\Gamma : s\Gamma \nrightarrow t\Gamma$ in $\mathcal{G}_{\mathrm{ML}}(\mathcal{T})$ corresponds to a list of dependent types in $\mathcal{T}$ and thus induces a dependent context
$$\vec{x}_s : s\Gamma^\circ, \vec{x}_t : t\Gamma^\circ \vdash \Gamma^\circ(\vec{x}_s, \vec{x}_t)$$
in $\mathcal{T}$.

- A 0-term in $f : \Gamma \to A$ in $\mathcal{G}_{\mathrm{ML}}(\mathcal{T})$ is a term

  $$\vec{x} : \Gamma^\circ \vdash f^\circ \vec{x} : A^\circ$$

  in $\mathcal{T}$.

- Suppose that Γ is the $(n+1)$-context in $\mathcal{G}_{\mathrm{ML}}(\mathcal{T})$ corresponding to the dependent context $\vec{x_s} : s\Gamma^\circ, \vec{x_t} : t\Gamma^\circ \vdash \Gamma(\vec{x}_s, \vec{x}_t)^\circ$ in $\mathcal{T}$. Then an $(n+1)$-term in $f : \Gamma \to A, \quad sf \nrightarrow tf$ in $\mathcal{G}_{\mathrm{ML}}(\mathcal{T})$ is a term

  $$\vec{x} : \Gamma(\vec{x}_s, \vec{x}_t)^\circ \vdash f^\circ \vec{x} : A^\circ((sf)^\circ(\vec{x_s}), (tf)^\circ(\vec{x_t}))$$

  in $\mathcal{T}$.

- It follows that each substitution $\Gamma \to \Delta$ in $\mathcal{G}_{\mathrm{ML}}(\mathcal{T})$ corresponds to a context morphism $\Gamma^\circ \to \Delta^\circ$ in $\mathcal{T}$. Hence, composition of terms in $\mathcal{G}_{\mathrm{ML}}(\mathcal{T})$ is defined by substitution in $\mathcal{T}$. The unitality and associativity of this composition follow from the unitality and associativity of the composition of context morphisms in $\mathcal{T}$.

2.4 The Span Construction

Batanin [6] describes the following class of examples:

Definition 2.4.0.1. Let $\mathcal{C}$ be a category with pullbacks. There is a globular multicategory $\mathrm{Span}\,\mathcal{C}$ such that:

- An n-type in $\mathrm{Span}\,\mathcal{C}$ is a functor $A : \mathrm{el}(n)^{\mathrm{op}} \to \mathcal{C}$ from the category of elements of the representable globular set n.

- It follows that, given a pasting diagram $\pi \in \mathbf{pd}(n)$, a context with shape π in $\mathrm{Span}\,\mathcal{C}$ amounts to a functor

  $$\Gamma : \mathrm{el}(\pi)^{\mathrm{op}} \longrightarrow \mathcal{C}.$$

  Associated to such a context there is a canonical functor $\Gamma' : \mathrm{el}(n)^{\mathrm{op}} \to A$, which sends an object of $\mathrm{el}(n)$, that is an arrow $s : k \to n \in \mathbb{G}_n$, to the limit of the following diagram:

  $$\pi_{\partial_k}^{\mathrm{op}} \xrightarrow{\mathrm{el}(\pi_s)^{\mathrm{op}}} \mathrm{el}(\pi)^{\mathrm{op}} \xrightarrow{\Gamma} \mathcal{C},$$

  and sends arrows in $\mathrm{el}(n)$ to the canonical morphisms induced between these limits.

- A term $f : \Gamma \to A$ in $\mathrm{Span}_n(\mathcal{C})$ is a natural transformation $\Gamma' \to A$.

Remark 2.4.0.2. We refer to functors $\mathrm{el}(n)^{\mathrm{op}} \to \mathcal{C}$ as *n-spans* in $\mathcal{C}$ (see [6]). The globular multicategory $\mathrm{Span}(\mathcal{C})$ underlies the monoidal globular category $\mathrm{Span}(\mathcal{C})$ described ibid. A 1-span is a span in the usual sense. More generally, an $(n+1)$-span is a "span between n-spans". For example, a 3-span is a diagram of the following form:

$$\begin{tikzcd}
& \bullet \arrow[dl] \arrow[dr] & \\
\bullet \arrow[d] \arrow[drr] & & \bullet \arrow[d] \arrow[dll] \\
\bullet \arrow[d] \arrow[drr] & & \bullet \arrow[d] \arrow[dll] \\
\bullet & & \bullet
\end{tikzcd}$$

Example 2.4.0.3. Suppose that $\mathcal{C}$ has finite limits. For any objects $A, B \in \mathcal{C}$, we have that $\mathrm{Span}(\mathcal{C})(A, B) = \mathrm{Span}(\mathcal{C}/\mathcal{A} \times \mathcal{B})$. Now suppose that $n > 0$, and that $M : A \nrightarrow B$ and $N : B \nrightarrow C$ are parallel n-types in $\mathrm{Span}\,\mathcal{C}$. Then by repeatedly taking pullbacks, we obtain an n-span $M \otimes_{n-1} N : A \nrightarrow C$. We have that $\mathrm{Span}(\mathcal{C})(M, N) = \mathrm{Span}(C/A \otimes_{n-1} B)$.

Remark 2.4.0.4. A particularly important case to consider is the globular multicategory Span(Set). This object plays the role of the "internal category of sets" in GlobMult. (See [51] and Example 2.9.3.8 below). We write

$$\mathrm{SpanSet} = \mathrm{Span}(\mathrm{Set}).$$

Example 2.4.0.5. In Chapter 4 we will frequently consider subobjects of globular multicategories of spans, whose n-types are spans which are *fibrations* in one sense or another.

Definition 2.4.0.6. For finite n, we define $\mathrm{Span}_n\,\mathcal{C} = \mathrm{tr}_n\,\mathrm{Span}\,\mathcal{C}$.

2.5 Globular Operads

Batanin's [6] globular operads are another important class of globular multicategories.

Definition 2.5.0.1. A *globular operad* is a globular multicategory $\mathbb{X}$ such that $\mathrm{Type}\,\mathbb{X} = \top$, the terminal globular set. In other words, a globular operad has a unique n-type for each $n \in \mathbb{G}$. When $\mathbb{X}$ is a globular operad, we denote the canonical n-type in $\mathbb{X}$ by n, and the canonical π-shaped n-context in $\mathbb{X}$ by π.

Example 2.5.0.2. Let $\mathbb{G}$ be the globular operad whose only terms are identity terms. We think of $\mathbb{G}$ as the *theory of globular sets*. See Example 2.6.0.3.

Example 2.5.0.3. When that $n = 1$, globular operads are the same as non-symmetric operads in the usual sense.

Example 2.5.0.4. The terminal globular multicategory, $\mathbb{1}$, is a globular operad and has a unique π-shaped n-term for each $\pi \in \mathbf{pd}(n)$. We think of $\mathbb{1}$ as the *theory of strict ω-categories*.

2.5.1 Contractible Operads

Definition 2.5.1.1. A *contraction* on a globular operad $\mathbb{P}$ consists of, for each n-pasting diagram π, and each pair of term-wise parallel π_∂-shaped n-terms $g, h : \pi_\partial \to n - 1$ in $\mathbb{P}$, a choice of n-term $\mathbf{l}_\pi^{g,h} : \pi \to n$ in $\mathbb{P}$ such that $\mathbf{l}_\pi^{g,h} : g \nrightarrow h$. We say that $\mathbb{P}$ is *contractible* when there exists a contraction on $\mathbb{P}$.

Remark 2.5.1.2. We adopt a slight variation of Leinster's notion of contraction (see [30], which includes a lifting condition for 0-terms. This notion can naturally be understood homotopically (see [22] and Section 5.3 of this thesis). A comparison between Leinster's notion, and Batanin's original notion [6] can be found in [16]

Definition 2.5.1.3. We say that a globular operad $\mathbb{P}$ is *normalised* when it has a unique 0-term, $\mathrm{id}_0 : 0 \to 0$.

Example 2.5.1.4. Various notions of weak higher category are parametrised by a normalised contractible globular operad. See for instance [6–8, 14, 16, 30].

2.5.2 Endomorphism Operads

Definition 2.5.2.1. A *globular object*, A, in a globular multicategory $\mathbb{X}$ consists of a 0-type A_0 in $\mathbb{X}$, together with, for each $1 \leq k$, a k-type $A_k : A_{k-1} \nrightarrow A_{k-1}$.

Example 2.5.2.2. Suppose that $\mathcal{C}$ is a category with pullbacks. Then a globular object A in $\mathrm{Span}\,\mathcal{C}$ is precisely a globular object $A : \mathbb{G}^{\mathrm{op}} \to \mathcal{C}$ in $\mathcal{C}$.

Let A be a globular object in a globular multicategory $\mathbb{X}$. Then for each pasting diagram π, there is a canonical context π_A such that for each k-cell $x \in \pi(k)$, we have that $\pi_A(x) = A_k$.

Remark 2.5.2.3 (see ([6])). Let A and $\mathbb{X}$ be as above. Then the *endomorphism operad* $\operatorname{End} A$ is the subobject of $\mathbb{X}$ such that a term $f : \pi \to n$ in $\operatorname{End} A$ is a π-shaped term

$$\pi_A \longrightarrow A_n$$

in $\mathbb{X}$. The assignment $A \mapsto \operatorname{End} A$ extends to arrows in an obvious way. In particular, whenever $\mathbb{X} = \operatorname{Span} \mathcal{C}$, this assignment defines the objects-part of a functor

$$\operatorname{End} : \mathcal{C}^{\mathbb{G}^{\mathrm{op}}} \longrightarrow \mathrm{GlobMult}$$

2.6 Algebras

Following a general trend in categorical semantics, we can view a globular multicategory as an *algebraic theory.* Under this lens, we make the following definition:

Definition 2.6.0.1. An *algebra* of a globular multicategory $\mathbb{X}$ is a homomorphism $\mathbb{X} \to \mathrm{SpanSet}$. A *homomorphism* between algebras is a transformation between these homomorphisms of globular multicategories.

Example 2.6.0.2. Algebras of the terminal globular operad, $\mathbb{1}$, are strict n-categories. When $\mathbb{P}$ is a normalised contractible operad, algebras of $\mathbb{P}$ are some sort of weak higher category.

Example 2.6.0.3. Algebras of the operad, $\mathbb{G}$, are globular sets.

Example 2.6.0.4. When $n = 1$, and $\mathbb{X}$ is an operad, we recover the usual notion of algebra of a non-symmetric operad.

Definition 2.6.0.5. Issues of size will not play a large rule in this thesis, but we will very occasionally need a good notion of *small set.* For concreteness, we define a small set to be a set within a particular Grothendieck universe. We say that a globular multicategory $\mathbb{X}$ is *small*, when $\operatorname{Type} \mathbb{X}(n)$ and $\operatorname{Term} \mathbb{X}(n)$ are small sets, for all n. We say that a homomorphism of globular categories is *small* when its type-wise and term-wise fibers are small sets.

Definition 2.6.0.6. Suppose that $\mathbb{X}_0$ and $\mathbb{X}$ are small globular multicategories, and that $\mathbb{I} : \mathbb{X}_0 \to \mathbb{X}$ is a homomorphism of globular multicategories. Then we have an adjunction of the following form:

$$\begin{tikzcd}
\mathrm{GlobMult}(\mathbb{X}_0, \mathrm{SpanSet}) \arrow[r, bend left, "\mathrm{Lan}_{\mathbb{I}}"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult}(\mathbb{X}, \mathrm{SpanSet}) \arrow[l, bend left, "\mathbb{I} \circ -"]
\end{tikzcd}$$

Proof. By construction, every n-context Γ in SpanSet can be viewed as an n-type Γ' taking pullbacks. Suppose that we have an algebra $\mathbb{F}_0 : \mathbb{X}_0 \to \text{SpanSet}$. Then we define the data of the homomorphism $\mathbb{F} = \text{Lan}_{\mathbb{I}}(\mathbb{F}_0)$ inductively so that:

- For each k-type $A_0 \in \mathbb{X}_0$, and each element $a_0 \in \mathbb{F}_0(A_0)$, there is an element
$$\iota(a_0) \in \mathbb{F}(\mathbb{I}(A_0)).$$
We require that $\iota(a_0) : \iota(sa_0) \twoheadrightarrow \iota(ta_0)$. It follows that for each k-context Γ, and each $\vec{a}_0 \in \Gamma'$, we have an element $\iota(\vec{a}_0) \in \mathbb{F}(\mathbb{I}(\Gamma))'$.
- For each k-term $f : \Gamma \to B$ in $\mathbb{X}$, and each $\vec{a} \in \mathbb{F}(\mathbb{I}(\Gamma))'$, there is an element
$$\mathbb{F}(f)(\vec{a}) \in \mathbb{F}(\mathbb{I}(B)).$$
We require that $F(f)(\vec{a}) : F(sf)(s\vec{a}) \twoheadrightarrow F(tf)(t\vec{a})$.
- For each term $f_0 : \Gamma_0 \to B_0$ in $\mathbb{X}_0$, and each element $a_0 \in \mathbb{F}_0(\Gamma)'$, we require that:
$$\mathbb{F}(\mathbb{I}(f_0))(\iota(\vec{a}_0)) = \iota(\mathbb{F}_0(f_0)(\vec{a}_0))$$
- For each type $A \in \mathbb{X}$, and each element $a \in \mathbb{F}(A)$, we require that:
$$\mathbb{F}(\text{id}_A)(a) = a.$$
- Suppose that we have $f : \Gamma \to \Delta$ and $g : \Delta \to B$ in $\mathbb{X}$. Then for each $\vec{a} \in \mathbb{F}(\Gamma)'$, we require that:
$$\mathbb{F}(f; g)(\vec{a}) = \mathbb{F}(g)(\mathbb{F}(f)(\vec{a})).$$

The size requirements placed on $\mathbb{X}$ ensure that this definition makes sense. The required universal property is easily verified. □

Remark 2.6.0.7. We can also see this result by noting that $\mathbb{I} : \mathbb{X}_0 \to \mathbb{X}$ corresponds to a map between essentially algebraic theories, and that algebras of $\mathbb{X}_0$ and $\mathbb{X}$ are models of these essentially algebraic theories.

Remark 2.6.0.8. In many cases, the adjunction defined by this left Kan extension is monadic. In particular, let $\mathbb{X}$ be any small globular multicategory, and let $\mathbb{X}_0$ be the globular multicategory with the same types as $\mathbb{X}$ but whose only terms are identity terms, and let $\mathbb{I} : \mathbb{X}_0 \to \mathbb{X}$ be the obvious inclusion. Then we have that

$$\text{GlobMult}(\mathbb{X}_0, \text{SpanSet}) = \mathbb{G}\text{-Set}/\,\text{Type}(\mathbb{X}_0)$$

Furthermore, algebras of $\mathbb{X}$ are the same as the algebras of the monad associated to $\mathbb{X}$ that is described by Leinster [30, §4.3], and the functor $\mathbb{I}\circ- :$ GlobMult$(\mathbb{X}, \text{SpanSet}) \to$ GlobMult$(\mathbb{X}_0, \text{SpanSet})$ is the monadic forgetful functor defined ibid.

Example 2.6.0.9. Let $\mathbb{X}$ be the globular multicategory inductively defined such that:

- There are two 0-types $A = \mathcal{H}^0 A$, and $B = \mathcal{H}^0 B$ in $\mathbb{X}$. A 0-term is either an identity term or a canonical term $f_0 : A \to B$.
- For each $k \geq 0$, there are exactly two $(k+1)$-types. These types are of the form:
$$\mathcal{H}^{k+1}A : \mathcal{H}^k A \nrightarrow \mathcal{H}^k A \qquad \mathcal{H}^{k+1}B : \mathcal{H}^k B \nrightarrow \mathcal{H}^k B$$
It follows that a π-shaped context Γ in $\mathbb{X}$, is completely determined by its source (or target) 0-type.
- Suppose that $n \geq 0$. Let Γ be an $(n+1)$-context in $\mathbb{X}$ and let M be a $(n+1)$-type in $\mathbb{X}$. We must have that $s\Gamma = t\Gamma$ and $sM = tM$. Suppose that we have an n-term $f_{s\Gamma,sM} : s\Gamma \to sM$. Then there is a unique $(n+1)$-term
$$f_{\Gamma,M} : \Gamma \longrightarrow M, \qquad f_{s\Gamma,sM} \nrightarrow f_{s\Gamma,sM}.$$

Thus, types and terms in $\mathbb{X}$ are completely determined by their 0-dimensional sources and terms are effectively "directed" from A to B. We say that a term $f : \Gamma \to M$ such that $s_0\Gamma = s_0M = A$ is in the A-*component* of $\mathbb{X}$. Similarly, we say that f is in the B-component of X when $s_0\Gamma = s_0M = B$. These components correspond to two canonical homomorphisms

$$\begin{tikzcd} \mathbb{1} \arrow[r, shift left, "\bar{A}"] \arrow[r, shift right, "\bar{B}"'] & \mathbb{X} \end{tikzcd}$$

from the terminal globular operad. Furthermore, to give an algebra $F : \mathbb{X} \to$ SpanSet is to give a pair of strict ω-categories $F\bar{A}, F\bar{B}$ together with a strict ω-functor $F\bar{A} \to F\bar{B}$ between them.

Let $\mathbb{X}_0$ be the subcategory of $\mathbb{X}$ whose 0-terms $f : \Gamma \to M$ either satisfy $s_0\Gamma = s_0M$ or have the form

$$f_{\mathcal{H}^k A, \mathcal{H}^k B} : \mathcal{H}^k A \to \mathcal{H}^k B$$

Then an algebra $\mathbb{F}_0 : \mathbb{X}_0 \to$ SpanSet amounts to a choice of strict ω-categories $\mathbb{F}_0\bar{A}, \mathbb{F}_0\bar{B}$ together with a map between their underlying globular sets. Let $\mathbb{I} : \mathbb{X}_0 \to \mathbb{X}$ be the obvious inclusion. Both left Kan extension and composition with $\mathbb{I}$ respect the source and target ω-categories, $\mathbb{F}\bar{A}$ and $\mathbb{F}\bar{B}$. Fixing $\mathbb{F}\bar{A}$ and $\mathbb{F}\bar{B}$, we obtain the adjunction

$$\begin{tikzcd}
\mathrm{Str}\,\omega\text{-}\mathrm{Cat}_{\mathrm{map}}(\mathbb{F}\bar{A},\mathbb{F}\bar{B}) \arrow[r, bend left, "\mathrm{Lan}_{\mathbb{I}}"] \arrow[r, phantom, "\perp"] & \mathrm{Str}\,\omega\text{-}\mathrm{Cat}(\mathbb{F}\bar{A},\mathbb{F}\bar{B}) \arrow[l, bend left, "\mathbb{I}\circ -"]
\end{tikzcd}$$

defining the free strict ω-functor monad on maps between the underlying globular sets of $F\bar{A}$ and $F\bar{B}$.

2.6.1 Discrete Opfibrations

Leinster [30] defines a *multicategory of elements* construction, associating, to each algebra $\mathbb{X} \to \mathrm{SpanSet}$, a *discrete opfibration* $\mathbb{Y} \to \mathbb{X}$. We will now describe this construction in our terminology and show how it can be seen as the result of pulling back along a *classifying discrete opfibration.*

Definition 2.6.1.1. We say that a homomorphism of globular multicategories $\mathbb{F} : \mathbb{Y} \to \mathbb{X}$ is a *discrete opfibration*, when for each context Γ in $\mathbb{Y}$ and each term $f : \mathbb{F}(\Gamma) \to A$ in $\mathbb{X}$, there is a unique term $\tilde{f} : \Gamma \to \tilde{A}$ such that $\mathbb{F}(\tilde{f}) = f$.

Remark 2.6.1.2. Leinster [30, §6.3] gives the following equivalent definition: a homomorphism of globular multicategories $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ is a discrete opfibration if the induced square

$$\begin{tikzcd}
& \mathrm{Term}\,\mathbb{X} \arrow[ld, "\mathrm{Ctx}"'] \arrow[dd, "\mathbb{F}_1"] \\
\mathbf{T}\,\mathrm{Type}\,\mathbb{X} \arrow[dd, "\mathbb{F}_0"'] & \\
& \mathrm{Term}\,\mathbb{Y} \arrow[ld, "\mathrm{Ctx}"] \\
\mathbf{T}\,\mathrm{Type}\,\mathbb{Y} &
\end{tikzcd}$$

is a pullback square.

Definition 2.6.1.3. Suppose that $\mathbb{X}$ is a globular multicategory, and that $\mathbb{F} : \mathbb{X} \to \mathrm{SpanSet}$ is an algebra of $\mathbb{X}$. Then we define $\mathrm{el}(\mathbb{F})$, the *globular multicategory of elements* of $\mathbb{F}$, as follows:

- An n-type in $\mathrm{el}(\mathbb{F})$ is a pair (A, a) where A is an n-type in $\mathbb{X}$, and $a \in \mathbb{F}A$. It follows that an n-context in $\mathrm{el}(\mathbb{F})$ amounts to a pair (Γ, γ), where Γ is an n-context, $\gamma \in \mathbb{F}(\Gamma)'$.

- An n-term $f : (\Gamma, \gamma) \to (A, a)$ in $\mathrm{el}(\mathbb{F})$ is an n-term in $f : \Gamma \to A$ in $\mathbb{X}$ such that $\mathbb{F}(f)(\gamma) = a$.

The next result now follows immediately:

Proposition 2.6.1.4. *The canonical projection* $\pi_{\mathbb{F}} : \mathrm{el}(\mathbb{F}) \to \mathbb{X}$ *is a discrete opfibration.*

Definition 2.6.1.5. We define $\mathrm{SpanSet}_\star$, the globular multicategory of pointed spans, as follows:

- A type of $\mathrm{SpanSet}_\star$ is a type $M : A \;\,{\to\!\!\!\!\!\to}\; B$ of SpanSet together with an element $m \in M$. It follows that a context Γ in $\mathrm{SpanSet}_\star$ amounts to a context Γ in SpanSet together with an element γ in the pullback Γ' defined by Γ.

- A term $f : (\Gamma, \gamma) \to (A, a), \quad sf \;\,{\to\!\!\!\!\!\to}\; tf$ of $\mathrm{SpanSet}_\star$ is a term $f : \Gamma \to A, \quad sf \;\,{\to\!\!\!\!\!\to}\; tf$ in SpanSet such that $f(\gamma) = a$.

Let $\pi_\star : \mathrm{SpanSet}_\star \to \mathrm{SpanSet}$ be the canonical projection. It is immediate from this definition that $\pi_\star$ is a discrete opfibration. Furthermore, our description of globular multicategories of elements makes the following alternative description evident:

Theorem 2.6.1.6. *The category of elements of an algebra* $\mathbb{F} : \mathbb{X} \to \mathrm{SpanSet}$ *is the pullback depicted in the following diagram:*

$$\begin{tikzcd}
\mathrm{el}(\mathbb{F}) \arrow[r] \arrow[d, "\pi_{\mathbb{F}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{SpanSet}_\star \arrow[d, "\pi_\star"] \\
\mathbb{X} \arrow[r, "\mathbb{F}"'] & \mathrm{SpanSet}
\end{tikzcd}$$

Remark 2.6.1.7. Leinster [30] observes that, for each globular multicategory $\mathbb{X}$, this defines an equivalence between the categories of small discrete opfibrations over $\mathbb{X}$ and algebras of $\mathbb{X}$. Thus, Theorem 2.6.1.6 says that $\pi_\star$ is a *classifying discrete opfibration*: small discrete opfibrations are precisely pullbacks of $\pi_\star$.

Remark 2.6.1.8. We are particularly interested in the case where $\mathbb{X}$ is a contractible globular operad. In this case, an algebra $\mathcal{C} : \mathbb{X} \to \mathrm{SpanSet}$ amounts to a weak higher category, and the multicategory of elements construction allows us to construct a globular multicategory from $\mathcal{C}$. Explicitly, a k-type in $\mathrm{el}(\mathcal{C})$ is a k-cell in the higher category $\mathcal{C}$, and a term $\tilde{f} : \Gamma \to A$ in $\mathrm{el}(\mathcal{C})$ amounts to a witness that the composite of the pasting diagram Γ, using some operation f in $\mathbb{X}$, is the cell A. It is notable that we do not need any restrictions at all on the weakness of $\mathcal{C}$ in order for this construction to work; whenever we have a weak ω-category described as an algebra, $\mathcal{C}$, of a globular operad, we have an n-globular multicategory, $\mathrm{el}(\mathcal{C})$.

Remark 2.6.1.9. An analogous analysis holds when $\mathcal{C} : \mathbb{P} \to \mathrm{SpanSet}_n$ is an n-category, for some finite n. In this case, the globular multicategory of elements of $\mathcal{C}$ is an n-globular multicategory $\mathrm{el}_n\,\mathbb{P}$.

2.7 The Vertical Construction

Suppose that $\mathbb{F} : \mathbb{P} \to \mathrm{SpanSet}$ is an algebra of a contractible globular operad $\mathbb{P}$. The globular multicategory of elements construction allows us to construct a globular multicategory whose *types* are cells in $\mathbb{F}$. In this section, we study a novel construction of a globular multicategory whose *terms* are cells in $\mathbb{F}$, given good conditions on $\mathbb{P}$.

Definition 2.7.0.1. We say that a normalised contractible globular operad $\mathbb{P}$ has *strict composition along* 0*-cells* when the following conditions hold:

- For each $k \geq 0$, and each $l \geq 0$, there is a canonical *compositor* k-term

  $$c^k_l : \underbrace{k \odot_0 k \odot_0 \cdots \odot_0 k}_{l \text{ times}} \longrightarrow k, \qquad c^{k-1}_l \twoheadrightarrow c^{k-1}_l.$$

  Here we take the 0-ary sum to be the 0-type 0. We require that compositors are closed under composition, and that $c^k_1 = \mathrm{id}_k$.

- **Interchange:** Suppose that π is a 0-trivial k-pasting diagram, and that $f : \pi \to k$ is a k-term in $\mathbb{P}$, for each $l > 0$, we have that

  $$\underbrace{(f \odot_0 f \odot_0 \cdots \odot_0 f)}_{l \text{ times}}; c^k_l = \left(\bigodot_{i\in\pi} c^{\dim i}_l\right); f.$$

  where $l \geq 0$, and f_i is a ρ_i-shaped pasting diagram for some ρ_i with a unique 0-component. When $l = 0$, we define the 0-ary sum on the left-hand side to be the unique 0-term $\mathrm{id}_\star : 0 \to 0$.

Example 2.7.0.2. The terminal globular operad $\mathbb{1}$ has strict composition along 0-cells.

Example 2.7.0.3. Whenever $n \geq 2$, and $\mathbb{P}$ is the weak n-category operad described by Batanin or Leinster, $\mathbb{P}$ does not have strict composition along 0-cells since composition of 1-cells is not strictly associative and unital.

Example 2.7.0.4. Let $\mathbb{P}$ be the 3-operad whose algebras are categories strictly enriched in the category of bicategories and strict 2-functors, with the cartesian product. Then $\mathbb{P}$ has strict composition along 0-cells.

In fact, we can generalise this last example in order to obtain a large class of contractible globular operads with strict composition along 0-cells.

Definition 2.7.0.5. Let $\mathbb{X}$ be a globular multicategory. Then we define the globular multicategory $\mathbf{E}(\mathbb{X})$ as follows:

- There is a unique 0-type $\star$ and a unique 0-term $\mathrm{id}_\star$.
- Suppose that $n > 0$ and that $\pi = \Sigma(\pi_1) \odot_0 \cdots \odot_0 \Sigma(\pi_l)$ is an n-pasting diagram. A π-shaped n-term in $\mathbf{E}(\mathbb{X})$ is a sequence of $(n-1)$-terms $f_1, \ldots f_l$ in $\mathbb{X}$ such that f_i is a π_i-shaped. Composition of n-terms in $\mathbf{E}(\mathbb{X})$ is induced by composition of $(n-1)$-terms in $\mathbb{X}$.

Remark 2.7.0.6. It is follows that an algebra of $\mathbf{E}(\mathbb{X})$ is precisely a category enriched in the category of algebras of $\mathbb{X}$ with its cartesian monoidal structure.

The following result now follows immediately from the definition of $\mathbf{E}$.

Theorem 2.7.0.7. *If $\mathbb{P}$ is a contractible globular operad, then $\mathbf{E}(\mathbb{P})$ is a normalised contractible globular operad with strict composition along 0-cells.*

Proof. Suppose that $\mathbb{P}$ is a contractible globular operad. Then $\mathbf{E}(\mathbb{P})$ is clearly normalised. When $k = 0$ we define c^0_l to be $\mathrm{id}_\star$. When $k > 0$, we define the compositor c^k_l to be the sequence $(\mathrm{id}_{k-1}, \cdots, \mathrm{id}_{k-1})$ of length l. The conditions on compositors are now easily verified.

We now verify contractibility of $\mathbf{E}(\mathbb{P})$. Suppose that $n \geq 0$, that π is an $(n+1)$-pasting diagram, and that $g, h : \pi_\partial \to n$ are term wise parallel n-terms in $\mathbf{E}(\mathbb{P})$. When $n = 0$, we must have that $g = h = \mathrm{id}_\star$, and

$$\pi = \underbrace{1 \odot_0 1 \odot_0 \cdots \odot_0 1}_{l \text{ times}} = \underbrace{\Sigma 0 \odot_0 \cdots \Sigma_0}_{l \text{ times}}.$$

Hence, we can define $f : \pi \to n+1, g \nrightarrow h$ by

$$f = \underbrace{(\mathrm{id}_0, \ldots, \mathrm{id}_0)}_{l \text{ times}}.$$

Suppose that $n > 0$, that $\pi = \Sigma(\pi_1) \odot_0 \cdots \odot_0 \Sigma(\pi_l)$. Then $\pi_\partial = \Sigma(\pi_1)_\partial \odot_0 \cdots \odot_0 \Sigma(\pi_l)_\partial$. Furthermore, we must have that

$$g = (g^1, \ldots, g^l), \qquad h = (h^1, \ldots, h^l),$$

where g, h are parallel $(\pi_i)_\partial$-shaped $(n-1)$-terms in $\mathbb{X}$. Hence, since $\mathbb{P}$ is contractible, for each $1 \leq i \leq l$, there exists $f^i : \pi_i \to n$, $g^i \nrightarrow h^i$ in $\mathbb{P}$. Hence, $f = (f^1, \ldots, f^l) : \pi \to n+1$ in $\mathbf{E}(\mathbb{P})$. Thus, $\mathbf{E}(\mathbb{P})$ is contractible. □

Remark 2.7.0.8. Suppose that $\mathbb{P}$ is a normalised contractible globular operad with strict composition along 0-cells. Let $\mathbb{F} : \mathbb{P} \to \mathrm{SpanSet}$ be an algebra of $\mathbb{P}$. Suppose that we have been given k-cells $f_1, \ldots f_l$ in $\mathbb{F}$ such that $t_0 f_i = s_0 f_{i+1}$. Then we define a notion of composition by:

$$f_1 \circ_0 f_2 \circ_0 \cdots f_l = \mathbb{F}(c_l^k)(f_1 \odot_0 f_2 \odot_0 \cdots \odot_0 f_l).$$

The properties of compositors ensure that the operation $- \circ_0 -$ is associative, and that the unit of $\circ_0$ at a 0-cell $A \in \mathbb{F}(0)$, is $\mathbb{F}(c_0^k)(A)$. Suppose that $g_1, \ldots, g_l$ are π-shaped diagrams in $\mathbb{F}$ such that $t_0 g_i = s_0 g_{i+1}$. Then we define the π-shaped diagram $g_1 \circ_0 \cdots \circ_0 g_l$ element-wise by setting

$$\begin{aligned} g_1 \circ_0 \cdots \circ_0 g_l &= \bigodot_{i \in \pi} (g_1)_i \circ_0 \cdots \circ_0 (g_l)_i \\ &= \bigodot_{i \in \pi} \mathbb{F}(c_l^{\dim i})((g_1)_i \odot_0 \cdots \odot_0 (g_l)_i) \\ &= \left(\bigodot_{i \in \pi} \mathbb{F}(c_l^{\dim i}) \right) \left(\bigodot_{i \in \pi} ((g_1)_i \circ_0 \cdots \circ_0 (g_l)_i) \right) \\ &= \mathbb{F}(\bigodot_{i \in \pi} c_l^{\dim i})(g_1 \odot_0 \cdots g_l). \end{aligned}$$

Suppose that π is a 0-trivial pasting diagram, and that $o : \pi \to k$ is a term in $\mathbb{P}$. Then **Interchange** implies that

$$\begin{aligned} \mathbb{F}(o)(g_1 \circ_0 \cdots \circ_0 g_l) &= \mathbb{F}(o)(\mathbb{F}(\bigodot_{i \in \pi} c_l^{\dim i})(g_1 \odot_0 \cdots \odot_0 g_l)) \\ &= \mathbb{F}(\left(\bigodot_{i \in \pi} c_l^{\dim i} \right); o)(g_1 \odot_0 \cdots \odot_0 g_l) \\ &= \mathbb{F}((o \odot_0 \cdots \odot_0 o); c_l^k)(g_1 \odot_0 \cdots g_l) \\ &= \mathbb{F}(o)(g_1) \circ_0 \mathbb{F}(o)(g_2) \cdots \circ_0 \mathbb{F}(o)(g_l) \end{aligned}$$

Definition 2.7.0.9. Suppose that $\mathbb{P}$ is a normalised contractible globular operad with strict composition along 0-cells. Let $\mathbb{F} : \mathbb{P} \to \mathrm{SpanSet}$ be an algebra of $\mathbb{P}$. Then we define the *vertical globular multicategory* $\mathbb{V}(\mathbb{F})$ as follows:

- A 0-type in A $\mathbb{V}(\mathbb{F})$ is a 0-cell $\bar{A}$ of $\mathbb{F}(0)$.
- A 0-term $f : A \to B$ in $\mathbb{V}(\mathbb{F})$ is a 1-cell $\bar{f} : \bar{A} \to \bar{B}$ in $\mathbb{F}(1)$. In this case, we define $\mathbf{o}_f$ to be the 1-term id_1 in $\mathbb{P}$.

- For each 0-cell $A \in \mathbb{F}$, and each $n > 0$, there is a unique n-type $\mathcal{H}^n_A$ such that $sH^n_A = tH^n_A = H^{n-1}A$. Given a 0-type A in $\mathbb{V}(\mathbb{F})$, we say that an n-context Γ is A-*simple*, when each type in Γ is of the form $\mathcal{H}^n_A$, for some $n \geq 0$.

- Suppose that $n > 0$. Given an A-simple π-shaped n-context Γ, an n-term $f : \Gamma \to \mathcal{H}^n_B$, $sf \ \twoheadrightarrow tf$ in $\mathbb{V}(\mathbb{F})$ consists of an $(n+1)$-cell

  $$\bar{f} : \overline{sf} \longrightarrow \overline{tf}$$

  in $\mathbb{F}$, together with a $(n+1)$-term

  $$\mathbf{o}_f : \Sigma\pi \longrightarrow n+1, \quad \mathbf{o}_{sf} \twoheadrightarrow \mathbf{o}_{tf}$$

  in $\mathbb{P}$. Now suppose that $f : \Gamma \to \Delta$ is a substitution. Then Γ must be A-simple, and Δ must be B-simple, for some 0-cells $A, B \in \mathbb{F}(0)$. Let $i \in \Sigma\Delta(n)$. Recall that when $n > 0$, each $i \in \Sigma\Delta(n)$ can be viewed as an element $i' \in \Delta(n-1)$. When $n = 0$, the set $\Sigma\Delta(n)$ has exactly two elements, $\star_s$, and $\star_t$, such that, for all $m > 0$, and all $j \in \Sigma\Delta(m)$, we have that $s_0 j = \star_s$ and $t_0 j = \star_t$. We define an n-cell $\bar{f}_i$ in $\mathbb{F}$ by:

  $$\bar{f}_i = \begin{cases} \overline{f_{i'}} & \text{if } n > 0 \\ A & \text{if } n = 0 \text{ and } i = \star_s \\ B & \text{if } n = 0 \text{ and } i = \star_t \end{cases}$$

  We define an n-term $\mathbf{o}^i_f$ in $\mathbb{P}$ by

  $$\bar{f}_i = \begin{cases} \mathbf{o}_{i'} & \text{if } n > 0 \\ \mathrm{id}_0 & \text{if } n = 0 \text{ and } i = \star_s \\ \mathrm{id}_0 & \text{if } n = 0 \text{ and } i = \star_t \end{cases}$$

  Hence we define

  $$\bar{f} = \bigodot_{i \in \Sigma\Delta} \overline{f_{i'}}, \qquad \mathbf{o}_f = \bigodot_{i \in \Sigma\Delta} \mathbf{o}_{f_i}.$$

  to be the corresponding $\Sigma\Delta$-shaped pasting diagrams in $\mathbb{F}$ and $\operatorname{Term}\mathbb{P}$ respectively,

- Suppose that we have an n-substitution $f : \Gamma \to \Delta$, and an n-term $g : \Delta \to A$ in $\mathbb{V}(\mathbb{F})$. Then we define the composite $f;g$ by:

  $$\overline{f;g} = \mathbb{F}(\mathbf{o}_g)(\bar{f}) \circ_0 \bar{g} \qquad \mathbf{o}_{f;g} = \mathbf{o}_f ; \mathbf{o}_g$$

- Composition is associative since

$$\begin{aligned}\overline{f;(g;h)} &= \mathbb{F}(\mathbf{o}_{g;h})(\bar{f}) \circ_0 \overline{g;h}\\ &= \mathbb{F}(\mathbf{o}_{g;h})(\bar{f}) \circ_0 \mathbb{F}(\mathbf{o}_h)(\bar{g}) \circ_0 \bar{h}\\ &= \mathbb{F}(\mathbf{o}_h)(\mathbb{F}(\mathbf{o}_g)\bar{f} \circ_0 \bar{g}) \circ_0 \bar{h}\\ &= \mathbb{F}(\mathbf{o}_h)(\overline{f;g}) \circ_0 \bar{h}\\ &= \overline{(f;g);h}\end{aligned}$$

  The middle equality follows from Remark 2.7.0.8.

- We define $\mathrm{id}_{\mathcal{H}^n_A}$ by

$$\overline{\mathrm{id}_{\mathcal{H}^n_A}} = \mathbb{F}(c_0^{n+1})(\bar{A}), \qquad \mathbf{o}_{\mathrm{id}_{\mathcal{H}^n_A}} = \mathrm{id}_{n+1}$$

- Composition is unital since, for any $f : \Gamma \to A$,

$$\begin{aligned}\overline{f;\mathrm{id}_A} &= \mathbb{F}(\mathrm{id}_{n+1})(\bar{f})\\ &= \bar{f}\\ &= \mathbb{F}(c_0^{n+1})(\bar{B}) \circ_0 \bar{f}\\ &= \mathbb{F}(\mathrm{id}_0; c_0^{n+1})(\bar{B}) \circ_0 \bar{f}\\ &= \mathbb{F}((\bigodot_{i\in\Sigma\Gamma} c_0^{\dim i}); \mathbf{o}_f)(\bar{B}) \circ_0 \bar{f}\\ &= \mathbb{F}(\mathbf{o}_f)(\mathbb{F}(\bigodot_{i\in\Sigma\Gamma} c_0^{\dim i})(\bar{B})) \circ_0 \bar{f}\\ &= \mathbb{F}(\mathbf{o}_f)(\mathbb{F}(\bigodot_{i\in\Gamma} c_0^{\dim i+1})(\bar{B})) \circ_0 \bar{f}\\ &= \overline{\mathrm{id}_\Gamma; f}\end{aligned}$$

  The middle equality follows from Remark 2.7.0.8.

Remark 2.7.0.10. An analogous analysis holds when $\mathcal{C} : \mathbb{P} \to \mathrm{SpanSet}_n$ is an n-category, for some finite n. In this case, the vertical globular multicategory $\mathbb{V}_n(\mathbb{X})$ is an $(n+1)$-globular multicategory.

Example 2.7.0.11. Suppose that $\mathcal{C}$ is a strict 2-category. Then there is a corresponding *vertical double category*. The virtual double category $\mathbb{V}(\mathcal{C})$ associated to this double category is such that:

- A 0-type in $\mathbb{V}(\mathcal{C})$ are objects of $\mathcal{C}$.
- A 0-term in $\mathbb{V}(\mathcal{C})$ is a 1-cell of $\mathcal{C}$.

- For each object $A \in \mathcal{C}$, there is a unique 1-type $H_A : A \nrightarrow A$ in $\mathbb{V}(\mathcal{C})$.

- A term of the form

$$\begin{tikzcd}
A \arrow[r, "H_A", "|" marking] \arrow[d, "f"'] & A \arrow[r, "H_A", "|" marking] \arrow[d, phantom, "\Downarrow\phi"] & A \arrow[d, "g"] \\
B \arrow[rr, "H_B"', "|" marking] & & B
\end{tikzcd}$$

  is a 2-cell $\phi : f \Rightarrow g$ in $\mathcal{C}$. The composite

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\psi"] & A \arrow[r, "H_A", "|" marking] \arrow[d, "i"] \arrow[dr, phantom, "\Downarrow\xi"] & A \arrow[d, "j"] \\
B \arrow[r, "H_B"', "|" marking] \arrow[d, "f"'] & B \arrow[r, "H_B"', "|" marking] \arrow[d, phantom, "\Downarrow\phi"] & B \arrow[d, "g"] \\
C \arrow[rr, "H_C"', "|" marking] & & C
\end{tikzcd}$$

  is the composite $(\psi \circ_1 \xi) \circ_0 \phi : f \circ_0 h \Rightarrow g \circ_0 j$. The coherence laws of $\mathcal{C}$ imply that this notion of composition is associative and unital.

2.8 Representability

Globular multicategories are close cousins of the *monoidal globular categories* introduced by Batanin [6] as a natural environment for studying higher categories. Every monoidal globular category has an underlying globular multicategory, and the globular multicategories arising in this way are characterised by a representability property. This correspondence is analogous to the characterization of monoidal categories as (non-symmetric) multicategories which are representable in a suitable sense. A very general statement of results of this flavour, including an unbiased variant of the results of this section, can be found in [17]. Here, we will explicitly describe the relationship between globular multicategories and monoidal globular categories.

Recall that a globular category $\mathcal{C}$ is a globular object in the category of categories. Whenever M is an object (or arrow) in $\mathcal{C}(k)$ such that $sM = A$ and $tM = B$, we write $M : A \nrightarrow B$, just as we do for globular multicategories. Whenever A is an object or arrow in $\mathcal{C}(0)$ we write $A : \star \nrightarrow \star$.

Definition 2.8.0.1 ([6]). A *monoidal globular category* is an ω-category internal to the category of globular categories whose unit associativity laws hold up to isomorphism. This amounts to a globular category $\mathcal{C}$ together with:

- for each $l < k$, *type-wise composition* functors

$$\otimes_l : \mathcal{C}(k) \times \mathcal{C}(k) \longrightarrow \mathcal{C}(k)$$

- for each k, a *unit* functor

$$\mathbf{Z} : \mathcal{C}(k) \longrightarrow \mathcal{C}(k+1)$$

- natural transformations and axioms, mimicking those of a strict ω-category up to isomorphism.

When these natural transformations are all identities, we say that a monoidal globular category is *strict*. We denote the category of monoidal globular categories by MonGlobCat.

Remark 2.8.0.2. Batanin [6] has proven a coherence theorem for monoidal globular categories: every monoidal globular category is equivalent to a strict monoidal globular category.

Remark 2.8.0.3. Monoidal n-globular categories are defined analogously.

Example 2.8.0.4. Suppose that $\mathcal{C}$ is a strict n-category. Then $\mathcal{C}$ can be seen as a strict monoidal $(n-1)$-globular category $\mathrm{Sq}\,\mathcal{C}$ such that:

- A k-type in $\mathrm{Sq}\,\mathcal{C}$ is a k-cell in $\mathcal{C}$.
- Type-wise composition comes from composition in $\mathcal{C}$.
- A k-term $f : M \to N, sf \nrightarrow tf$ in $\mathrm{Sq}\,\mathcal{C}$ is a $(k+1)$-cell:

$$\begin{tikzcd} A \arrow[r, "M"] \arrow[d, "sf"'] \arrow[dr, Rightarrow] & B \arrow[d, "tf"] \\ C \arrow[r, "N"'] & D \end{tikzcd}$$

- Composition of terms also comes from composition in $\mathcal{C}$.

Proposition 2.8.0.5. *There is a functor* $U_\otimes$: MonGlobCat $\to$ GlobMult. *This functor is injective on objects and faithful.*

Proof. Let $\mathcal{C}$ be a monoidal globular category. We define the globular multicategory $U_\otimes \mathcal{C}$ as follows:

- A k-type $M : A \nrightarrow B$ in $U_\otimes\mathcal{C}$ is an object $\ulcorner M \urcorner : \ulcorner A \urcorner \nrightarrow \ulcorner B \urcorner$ of $\mathcal{C}(k)$.

- Let $\Gamma = \bigodot_{x \in \Gamma} \Gamma_x$ be a k-context in $U_\otimes\mathcal{C}$. Repeatedly applying the type-wise composition functors $- \otimes -$ of $\mathcal{C}$, we obtain an object

  $$\ulcorner \Gamma \urcorner = \bigotimes_{x \in \Gamma} \mathbf{Z}^{k - \dim x} \ulcorner \Gamma_x \urcorner$$

  of $\mathcal{C}(k)$ such that $\ulcorner \Gamma \urcorner : \ulcorner s\Gamma \urcorner \nrightarrow \ulcorner t\Gamma \urcorner$. Here we need to choose an order of composition. However, this choice is unique up to canonical isomorphism because of the coherence theorem for monoidal globular categories.

- A k-term $f : \Gamma \to M, sf \nrightarrow tf$ in $U_\otimes C$ is a morphism

  $$\ulcorner f \urcorner : \ulcorner \Gamma \urcorner \longrightarrow \ulcorner M \urcorner, \qquad \ulcorner sf \urcorner \longrightarrow\!\!\!\!\!\!+\ \ulcorner tf \urcorner$$

  in $\mathcal{C}(k)$.

- Suppose that $g : \Delta \to \Gamma$ is a k-substitution in $U_\otimes\mathcal{C}$. Then the coherence axioms of $\mathcal{C}$ induce a canonical isomorphism,

  $$\ulcorner \Delta \urcorner = \bigotimes_{y \in \Delta} \Delta_y \xrightarrow[\sim]{\alpha} \bigotimes_{x \in \Gamma} \bigotimes_{y \in \Delta_x} \mathbf{Z}^{k - \dim x} \ulcorner (\Delta_x)_y \urcorner = \bigotimes_{x \in \Gamma} \ulcorner \Delta_x \urcorner.$$

  We define $\ulcorner g \urcorner : \ulcorner \Delta \urcorner \to \ulcorner \Gamma \urcorner$, $\ulcorner sg \urcorner \nrightarrow \ulcorner tg \urcorner$ to be the following composite:

  $$\ulcorner \Delta \urcorner \xrightarrow[\sim]{\alpha} \bigotimes_{x \in \Gamma} \ulcorner \Delta_x \urcorner \xrightarrow{\bigotimes_{x \in \Gamma} \ulcorner g_x \urcorner} \bigotimes_{x \in \Gamma} \ulcorner \Gamma_x \urcorner = \ulcorner \Gamma \urcorner.$$

- Whenever $g : \Delta \to \Gamma$ is a substitution in $U_\otimes\mathcal{C}$, and $f : \Gamma \to M$ is a term, we define

  $$\ulcorner f ; g \urcorner = \ulcorner f \urcorner ; \ulcorner g \urcorner, \qquad \text{and} \qquad \ulcorner \mathrm{id}_f \urcorner = \mathrm{id}_{\ulcorner f \urcorner}.$$

  The coherence laws of $\mathcal{C}$ ensure that composition of terms in $U_\otimes\mathcal{C}$ is associative and unital.

This assignment is easily seen to be functorial, injective on objects and faithful. □

Suppose that we have a globular multicategory of the form $U_\otimes\mathcal{C}$. Then we can recover the type-wise composition of $\mathcal{C}$ by looking for terms satisfying the following universal property:

Definition 2.8.0.6. We say that an n-substitution $f : \Gamma \to \Delta,\ sf \nrightarrow tf$ in a globular multicategory $\mathbb{X}$ is *strictly representing* when, for any $m \geq n$, any m-type A, and any term-wise parallel $(m-1)$-terms $g_s : s\Delta \to tA$ and $g_t : t\Delta \to tA$, the map

$$[\Delta \to A, \quad g_s \nrightarrow g_t] \xrightarrow{f;-} [\Gamma \to A, \quad sf;g_s \nrightarrow tf;g_t]$$

is a bijection.

Example 2.8.0.7. Suppose that $\mathbb{X} = U_\otimes \mathcal{C}$. Let $n \geq 1$, and let $M : A \nrightarrow B$ and $N : B \nrightarrow C$ be n-types in $\mathbb{X}$. Define the n-type $M \otimes_{n-1} N$ in $U_\otimes \mathcal{C}$ by

$$\ulcorner M \otimes_{n-1} N \urcorner = \ulcorner M \urcorner \otimes_{n-1} \ulcorner N \urcorner$$

Then the morphism $\mathrm{id}_{\ulcorner M \urcorner \otimes_{n-1} \ulcorner N \urcorner}$ in $\mathcal{C}$ corresponds to a term $m : M \odot_{n-1} N \to M \otimes_{n-1} N,\ \mathrm{id}_A \nrightarrow \mathrm{id}_C$ in $U_\otimes \mathcal{C}$. We think of m as witnessing the type-wise composition of M and N. Since composition with $\mathrm{id}_{M \otimes_{n-1} N}$ is a bijection in $\mathcal{C}$, the term m is strictly representing in $U_\otimes \mathcal{C}$.

This example motivates the following definition:

Definition 2.8.0.8. Suppose that Γ is an n-context in a globular multicategory $\mathbb{X}$. We say that a strictly representing n-term $\mathbf{m}_\Gamma : \Gamma \to \bigotimes \Gamma$ is a *compositor* for Γ if:

- We have that $n = 0$, and $\mathbf{m}_\Gamma = \mathrm{id}_\Gamma$.
- We have that $n > 1$, and $\mathbf{m}_\Gamma : \mathbf{m}_{s\Gamma} \nrightarrow \mathbf{m}_{t\Gamma}$, where $\mathbf{m}_{s\Gamma}$, $\mathbf{m}_{t\Gamma}$ are compositors for $s\Gamma$, $t\Gamma$ respectively.

In this case, we refer to $\bigotimes \Gamma$ as the *composite* of Γ.

Remark 2.8.0.9. It follows that composites are well-defined up to unique isomorphism.

Arguing as in Example 2.8.0.7, it is clear that, for any monoidal globular category $\mathcal{C}$, every context Γ in $U_\otimes \mathcal{C}$ has a compositor. In fact, this property characterises globular multicategories of this form.

Definition 2.8.0.10. A globular multicategory $\mathbb{X}$ is *representable* if and only if each context in $\mathbb{X}$ admits a compositor.

Proposition 2.8.0.11. *A globular multicategory is in the essential image of* $U_\otimes : \mathrm{MonGlobCat} \to \mathrm{GlobMult}$ *if and only if it is representable.*

Proof. We have already seen the "if"-direction. For the "only if"-direction, suppose that $\mathbb{X}$ is a globular multicategory, and that each context Γ in $\mathbb{X}$ admits a compositor $m_\Gamma : \Gamma \to \bigotimes \Gamma$. Then we define a monoidal globular category $\mathcal{C}$ such that:

- An object of $\mathcal{C}(n)$ is a type of $\mathbb{X}$.
- A morphism $f : A \to B$ of $\mathcal{C}(n)$ is a term $f : A \to B$ of $\mathbb{X}$.
- The type-wise composite of n-types $M : A \nrightarrow B$ and $N : B \nrightarrow C$ is defined to be the target $\bigotimes(A \odot B)$ of the compositor of the n-context $A \odot B$.
- The unit of an n-type A is defined to be the target $\bigotimes[A]$ of the compositor of the $(n+1)$-context $[A]$.

The coherence laws of $\mathcal{C}$ now follow from the universal properties of the compositors, and by construction we have that $U_\otimes \mathcal{C} \cong \mathbb{X}$. □

Example 2.8.0.12. A virtual double category is representable exactly when it underlies a pseudo-double category. See [17].

Example 2.8.0.13. Type-wise composites in Span $\mathcal{C}$ can be computed as certain limits; this essentially follows from the definition of Span. The unit of an n-type A in Span $\mathcal{C}$ is the $(n+1)$-span $A : A \nrightarrow A$ whose left and right legs are both identity arrows.

Remark 2.8.0.14. Suppose that $\mathbb{X}$ is a representable globular multicategory. Then terms in $\mathbb{X}$ can be composed type-wise. Suppose that $f : \Gamma \to \Delta$, $sf \nrightarrow tf$ is a substitution in $\mathbb{X}$. Let $\mathbf{m}_\Gamma, \mathbf{m}_\Delta$ be compositors of Γ and Δ respectively. Then the universal property of $\mathbf{m}_\Gamma$ implies that there exists a unique

$$\bigotimes f : \bigotimes \Gamma \longrightarrow \bigotimes \Delta, \qquad \bigotimes sf \nrightarrow \bigotimes tf,$$

such that $\mathbf{m}_\Gamma ; \bigotimes f = f ; \mathbf{m}_\Delta$.

Example 2.8.0.15. When $f : \Gamma \to \Delta$ is a substitution is Span $\mathcal{C}$, the composite $\bigotimes f : \bigotimes \Gamma \to \bigotimes \Delta$ is the following canonical arrow between limits

$$\bigotimes \Gamma = \lim_{y \in \Delta} \Gamma_y \xrightarrow{\lim f_y} \lim_{y \in \Delta} \Delta_y = \bigotimes \Delta.$$

From this point onwards, we identify MonGlobCat with the 2-category $\text{GlobMult}_\otimes$ of globular multicategories, with chosen compositors, and compositor preserving homomorphisms.

2.9 Families Constructions

In this section, we generalise the families construction on categories (see [9]) to a family (pun intended!) of constructions on globular multicategories. We will see that these constructions freely add coproducts, in a suitable sense.

We will need to consider the differences between ω-globular multicategories and finite dimensional globular multicategories more carefully in this section. Hence, for the remainder of this section, we suppose that $\mathbb{X}$ is a $\mathbf{d}$-globular multicategory for some $0 \leq \mathbf{d} \leq \omega$. For convenience of notation, we define $[\mathbf{d}] = \{n \text{ finite} \mid n \leq \mathbf{d}\}$.

2.9.1 The Total Families Construction

Definition 2.9.1.1. There is a $\mathbf{d}$-globular multicategory $\mathrm{Fam}_{\mathbf{d}}^{[\mathbf{d}]}\,\mathbb{X}$ whose types are indexed collections of types in $\mathbb{X}$. The assignment $\mathrm{Fam}_{\mathbf{d}}^{[\mathbf{d}]} : \mathrm{GlobMult} \to \mathrm{GlobMult}$ is strictly 2-functorial. We refer to this functor as the *total families construction.*

Proof. Let $\star$ be the unique (-1)-type of $\mathrm{Fam}_{\mathbf{d}}^{[\mathbf{d}]}\,\mathbb{X}$. Then we define $\mathbf{I}_\star$ to be the one-element set $\{\star\}$. We will define $\mathbf{I}_{\mathrm{id}_\star}$ to be the function $\mathrm{id}_\star : \{\star\} \to \{\star\}$. We now define the n-types, n-terms, and various related data of $\mathrm{Fam}_{\mathbf{d}}^{[\mathbf{d}]}\,\mathbb{X}$, for each $n \in [\mathbf{d}]$, by induction on n.

- An n-type $M : A \;\;\!\!\!\rightarrow\!\!\!\!\!\;+\; B$ in $\mathrm{Fam}_{\mathbf{d}}^{[\mathbf{d}]}\,\mathbb{X}$ consists of, for each $i \in \mathbf{I}_A$ and $j \in \mathbf{I}_B$, an index set $\mathbf{I}_M(i,j)$ together with, for each $k \in \mathbf{I}_M(i,j)$, an n-type

  $$M(k) : A(i) \longrightarrow\!\!\!\!\!\!+\;\; B(j).$$

  Note that $\mathbf{I}_M$ can equivalently be seen as an n-span of sets

  $$\mathbf{I}_M : \mathbf{I}_A \longrightarrow\!\!\!\!\!\!+\;\; \mathbf{I}_B.$$

- Suppose that $\Gamma : s\Gamma \;\;\!\!\!\rightarrow\!\!\!\!\!\;+\; t\Gamma$ is a π-shaped n-context in $\mathrm{Fam}_{\mathbf{d}}^{[\mathbf{d}]}\,\mathbb{X}$. Then we have a π-shaped diagram $(\mathbf{I}_{\Gamma_x})_{x\in\Gamma}$ in Type(SpanSet). Let $\mathbf{I}_\Gamma : \mathbf{I}_{s\Gamma} \;\;\!\!\!\rightarrow\!\!\!\!\!\;+\; \mathbf{I}_{t\Gamma}$ be the type-wise composite of this diagram; that is, $\mathbf{I}_\Gamma = \lim_{x\in\Gamma} \mathbf{I}_{\Gamma_x}$. For each $y \in \Gamma$, let $\pi_y : I_\Gamma \to I_{\Gamma_y}$ be the canonical projection from this limit. Let

  $$\mathbf{I}_{s\Gamma} \otimes_{n-1} \mathbf{I}_{t\Gamma} = \{i \in \mathbf{I}_{s\Gamma}, j \in \mathbf{I}_{t\Gamma} \mid i, j \text{ parallel in } \mathrm{Type(SpanSet)}\}$$

  For each $(i,j) \in \mathbf{I}_{s\Gamma} \otimes_{n-1} \mathbf{I}_{t\Gamma}$, let $\mathbf{I}_\Gamma(i,j)$ be the collection of elements in the set $\mathbf{I}_\Gamma$ mapping down to (i,j) in $\mathbf{I}_{s\Gamma}\otimes_{n-1}$. It follows that each $k \in \mathbf{I}_\Gamma(i,j)$ induces a π-shaped n-context $\Gamma(k) : s\Gamma(i) \;\;\!\!\!\rightarrow\!\!\!\!\!\;+\; t\Gamma(j)$.

- An n-term $f : \Gamma \to M, sf \nrightarrow tf$ in $\mathrm{Fam}^{[\mathbf{d}]}_{\mathbf{d}}\,\mathbb{X}$ consists of, for each $i \in \mathbf{I}_{s\Gamma}$ and each $j \in \mathbf{I}_{t\Gamma}$, a function

  $$\mathbf{I}_f : \mathbf{I}_\Gamma(i,j) \to \mathbf{I}_M(sf(i), tf(j)),$$

  together with a term

  $$f(k) : \Gamma(k) \longrightarrow M(\mathbf{I}_f(k)), \quad sf(i) \longrightarrow\!\!\!\!\!\!\nrightarrow tf(j)$$

  for each $k \in \mathbf{I}_\Gamma(i,j)$.

- Suppose that $f : \Gamma \to \Delta$ is an n-substitution in $\mathrm{Fam}^{[\mathbf{d}]}_{\mathbf{d}}\,\mathbb{X}$. We define $\mathbf{I}_f$ to be the canonical function

  $$\mathbf{I}_\Gamma = \lim_{y \in \Delta} \mathbf{I}_{\Gamma_y} \xrightarrow{\lim \mathbf{I}_{f_y}} \lim_{y \in \Delta} \mathbf{I}_y = \mathbf{I}_\Delta.$$

  For each $i \in \mathbf{I}_{s\Gamma}, j \in \mathbf{I}_{t\Gamma}$ and $k \in \mathbf{I}_\Gamma(i,j)$, we define

  $$f(k) = \bigodot_{y \in \Delta} f_y(\pi_y k).$$

  By construction, we have that $f(k) : \Gamma(k) \to \Delta(\mathbf{I}_f(k)), sf(i)) \nrightarrow tf(j)$. Now suppose that we have a term $g : \Delta \to M$ in $\mathrm{Fam}^{[\mathbf{d}]}_{\mathbf{d}}\,\mathbb{X}$. Then, for each i, j, k as above, we define $\mathbf{I}_{f;g}(k) = \mathbf{I}_g(\mathbf{I}_f(k))$ and

  $$(f;g)(k) = f(k); g(\mathbf{I}_f(k)) \quad : \quad \Gamma(k) \nrightarrow M(\mathbf{I}_{f;g}(k)), \quad s(f;g) \nrightarrow t(f;g).$$

- Since for any n-type M, we have that $\bigodot[M] = M$, we have that $\mathbf{I}_{[M]} = \mathbf{I}_M$. Hence, we define the unit term $\mathrm{id}_M : [M] \to M$ by

  $$\mathbf{I}_{\mathrm{id}_M} = \mathrm{id}_{\mathbf{I}_M}, \qquad \mathrm{id}_M(i) = \mathrm{id}_{M(i)}\,.$$

  The associativity and the unit laws now follow from the corresponding laws in $\mathbb{X}$.

It is easily verified that this assignment is strictly 2-functorial. ☐

Example 2.9.1.2. Consider the terminal n-globular multicategory $\mathbb{1}_n$. Then $\mathrm{Fam}^{[n]}_n(\mathbb{1}_n) \cong \mathrm{Span}_n(\mathrm{Set})$.

Example 2.9.1.3. A 0-globular multicategory is just a category $\mathcal{C}$, and $\mathrm{Fam}^{[0]}_0\,\mathcal{C}$ is the usual families construction on categories (see [9]).

Remark 2.9.1.4. Truncation commutes with the total family construction; for any $n \in [\mathbf{d}]$, we have a natural isomorphism $\mathrm{tr}_n \operatorname{Fam}_{\mathbf{d}}^{[\mathbf{d}]} \mathbb{X} \cong \operatorname{Fam}_n^{[n]} \mathrm{tr}_n \mathbb{X}$.

Proposition 2.9.1.5. *Suppose that $\mathbb{X}$ is representable. Then $\operatorname{Fam}_{\mathbf{d}}^{[\mathbf{d}]} \mathbb{X}$ is also representable.*

Proof. We define n-compositors in $\operatorname{Fam}_{\mathbf{d}}^{[\mathbf{d}]} \mathbb{X}$ by induction on $n \geq 0$. Suppose that Γ is an n-context. If $n = 0$, then $\bigotimes \Gamma = \Gamma$ and we define $\mathbf{m}_\Gamma = \mathrm{id}_\Gamma$. Suppose that $n \geq 1$. and that Γ is an n-context in $\operatorname{Fam}_{\mathbf{d}}^{[\mathbf{d}]} X$. Then we define the n-type $\bigotimes \Gamma$ by:

$$\begin{aligned} \mathbf{I}_{\bigotimes \Gamma} &= \bigotimes (\mathbf{I}_x)_{x \in \Gamma} &&= \mathbf{I}_\Gamma \\ (\bigotimes \Gamma)(k) &= \bigotimes (A(k))_{x:A \in \Gamma} &&= \bigotimes \Gamma(k) \end{aligned}$$

We define the compositor $\mathbf{m}_\Gamma : \Gamma \to \bigotimes \Gamma$, $\mathbf{m}_{s\Gamma} \;\nrightarrow\; \mathbf{m}_{t\Gamma}$ by setting, for each $k \in \mathbf{I}_\Gamma(i,j)$,

$$\begin{aligned} \mathbf{I}_{\mathbf{m}_\Gamma}(k) &= \bigotimes (\mathbf{m}_{\mathbf{I}_A})_{x:A \in \Gamma} \\ \mathbf{m}_\Gamma(k) &= \mathbf{m}_{\Gamma(k)} \end{aligned}$$

Now suppose that $f : \Gamma \to M$, $\mathbf{m}_{s\Gamma}; g_s \;\nrightarrow\; \mathbf{m}_{t\Gamma}; g_t$ is an m-term in $\operatorname{Fam}_{\mathbf{d}}^{[\mathbf{d}]} \mathbb{X}$, for some $m \geq n$. Then we define $g : \bigotimes \Gamma \to M$, $g_s \;\nrightarrow\; g_t$ by setting $\mathbf{I}_g = \mathbf{I}_f$, and, for each $k \in \mathbf{I}_\Gamma(i,j)$, defining $g(k) : \bigotimes \Gamma \to M$, $g_s(i) \;\nrightarrow\; g_t(i)$ to be the unique term such that $\mathbf{m}_\Gamma; g(k) = f(k)$. It follows that $m_\Gamma; g = f$ and that g is the unique term in $\operatorname{Fam}_{\mathbf{d}}^{[\mathbf{d}]} \mathbb{X}$ with this property. □

2.9.2 Level-wise Families Constructions

Frequently, we only consider families at a particular level (dimension) or set of levels of $\mathbb{X}$.

Definition 2.9.2.1. Let $S \subseteq [\mathbf{d}]$. Then we define $\operatorname{Fam}_{\mathbf{d}}^{S} \mathbb{X}$ to be the subobject of $\operatorname{Fam}_{\mathbf{d}}^{[\mathbf{d}]} \mathbb{X}$ such that:

- Suppose that $n \notin S$ and that $M : A \;\nrightarrow\; B$ is an n-type in $\operatorname{Fam}^s \mathbb{X}$. We require the following diagram to be a pullback square:

$$\begin{tikzcd} \mathbf{I}_M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{I}_A \arrow[d] \\ \mathbf{I}_B \arrow[r] & \mathbf{I}_{\partial^2 M} \end{tikzcd}$$

  , where $\mathbf{I}_{\partial^2 M} = \mathbf{I}_{sA} \otimes_{n-1} \mathbf{I}_{tA} = \mathbf{I}_{sB} \otimes_{n-1} \mathbf{I}_{tB}$. In other words, for all $(i,j) \in \mathbf{I}_{\partial M}$, we require that $\mathbf{I}_M(i,j) = \{\star\}$. In particular when $n < \operatorname{Min} S$, we have that $\mathbf{I}_M = \{\star\}$. Thus, in this case, an n-type in $\operatorname{Fam}_{\mathbf{d}}^{S} \mathbb{X}$ can equivalently be seen as an n-type in $\mathbb{X}$.

- When $n \in S$, we do not impose any further restrictions on n-types or n-terms (besides the restrictions on their source and target types and terms).

Remark 2.9.2.2. In the infinite dimensional case, we define

$$\mathrm{Fam}^S \mathbb{X} = \mathrm{Fam}^S_\omega S.$$

Example 2.9.2.3. Suppose that $\mathbb{X}$ is a 1-globular multicategory with a unique 0-type. Then $\mathbb{X}$ can be seen an ordinary multicategory, and $\mathrm{Fam}^{\{0\}}_1 \mathbb{X}$ is the matrices construction described by Leinster [30].

Remark 2.9.2.4. Truncation commutes with level-wise family constructions: whenever $n \leq \mathbf{d}$ and $S \subseteq [n]$, we have a natural isomorphism

$$\mathrm{tr}_n \, \mathrm{Fam}^S_{\mathbf{d}} \mathbb{X} \cong \mathrm{Fam}^S_n \, \mathrm{tr}_n \mathbb{X}.$$

Remark 2.9.2.5. Level-wise families constructions do *not* necessarily preserve representability. Suppose that $\mathbf{d} = 1$. Then every monoidal category $(\mathcal{C}, \otimes, I)$ can be viewed as a representable 1-globular multicategory, with a unique 0-type, and whose 1-types are the objects of $\mathcal{C}$. Let A be a set; that is, a 0-type in $\mathrm{Fam}^{\{0\}}_1(\mathcal{C})$. Then, following [17], when $\mathcal{C}$ has small coproducts, and $\otimes$ preserves them on both sides, the globular multicategory $\mathrm{Fam}^{\{0\}}_1(\mathcal{C})$ is representable. In particular, when $\mathcal{C}$ has an initial object $\perp$ and $\otimes$ preserves initial objects, we may define a compositor, $\mathbf{m}_{[A]} : [A] \to \mathcal{H}_A, \; \mathrm{id}_A \;\nrightarrow \mathrm{id}_A$, of the 1-context $[A]$ by

$$\mathcal{H}_A(a, a') = \begin{cases} I & \text{if } a = a' \\ \perp & \text{if } a \neq a' \end{cases} \qquad m_{[A]}(a) = \mathrm{id}_I \, .$$

On the other hand, suppose that $\mathcal{C}$ does not have an initial object. Then $\mathrm{Fam}^{\{0\}}_1(\mathcal{C})$ need not be representable. For example, suppose that $\mathcal{C}$ is the category of non-empty sets with the cartesian product as its monoidal structure. Suppose that A is a two-element set, and let M be the 1-type in $\mathrm{Fam}^{\{0\}}_1(\mathcal{C})$ defined by:

$$M(a, a') = \begin{cases} \{\star\} & \text{if } a = a' \\ A & \text{if } a \neq a' \end{cases}$$

Then, there is a unique term $A \to M, \; \mathrm{id}_A \;\nrightarrow \mathrm{id}_A$. However, for any 1-type $N : A \nrightarrow A$, considering $N(a, a')$, for $a \neq a'$, we find that there must be at least four different terms in $[N \to M, \quad \mathrm{id}_A \;\nrightarrow \mathrm{id}_A]$. Hence, the 1-context $[A]$ cannot have a compositor in $\mathrm{Fam}^{\{0\}}_1(\mathcal{C})$.

However, level-wise families constructions do preserve a weaker notion of representability.

Definition 2.9.2.6. We say that a $\mathbf{d}$-globular category is *representable up to level* $l \in [\mathbf{d}]$, when every l'-context has a compositor for $l' \leq l$.

Proposition 2.9.2.7. *Suppose that the globular multicategory* $\mathbb{X}$ *is representable up to level* l*. Then* $\mathrm{Fam}_n^{\{l\}}(\mathbb{X})$ *is representable up to level* l*.*

Proof. Suppose that $l' \leq l$, and that Γ is an l'-context in $\mathrm{Fam}_{\mathbf{d}}^{\{l\}}(\mathbb{X})$. Then the compositor of Γ described in Proposition 2.9.1.5 is a compositor in $\mathrm{Fam}_{\mathbf{d}}^{\{l\}}(\mathbb{X})$. ☐

2.9.3 Coproducts

It is well known (see for instance [9][(3.5)]) that the families construction freely adds small coproducts to categories. We can characterise the total and level-wise families constructions by similar properties.

Definition 2.9.3.1. Let $\mathbf{A} = \{A_i : sA \nrightarrow tA\}_{i \in I}$ be a collection of parallel l-types in $\mathbb{X}$. A *coproduct* of $\mathbf{A}$ consists of the following data:

- An l-type $\coprod_{i \in I} A_I : sA \nrightarrow tA$, together with, for each $i \in I$, an inclusion term

$$\iota_i : A_i \to \coprod_{i \in I} A_i, \quad \mathrm{id}_{sA} \nrightarrow \mathrm{id}_{tA} .$$

- Suppose that $n \geq l$. Suppose that Γ is a π-shaped n-context in $\mathbb{X}$ such that, for some l-variable x in Γ, we have that $\Gamma_x = \coprod_{i \in I} A_I$. Suppose that x is not the source or target of any other variable in Γ. Then, for each i, since $\coprod_{i \in I} A_I$ and A_i are parallel, there is a context $\Gamma[A_i/x]$, together with an inclusion term $\iota_i^\Gamma : \Gamma[A_i/x] \to \Gamma$ defined by

$$\Gamma[A_i/x](y) = \begin{cases} A_i & \text{if } y = x \\ \Gamma_y & \text{if } y \neq x \end{cases} \qquad \iota_i^\Gamma = \begin{cases} \iota^i & \text{if } y = x \\ \mathrm{id}_{\Gamma_y} & \text{if } y \neq x \end{cases}$$

  for each $y \in \pi$. When $n = l$, we have that $\iota_i^\Gamma : \mathrm{id}_{s\Gamma} \nrightarrow \mathrm{id}_{t\Gamma}$, and when $n > l$, we have that $\iota_i^\Gamma : \iota_i^{s\Gamma} \nrightarrow \iota_i^{t\Gamma}$. For any n-type B in X, we require that the induced function

$$[\Gamma \to B, \quad g \nrightarrow h] \xrightarrow{(\iota_i^\Gamma)_{i \in I};-} \prod_{i \in I} [\Gamma[A_i/x] \to B, \quad s_{n-1}\iota_i^\Gamma ; g \nrightarrow t_{n-1}\iota_i^\Gamma ; h]$$

  is a bijection.

We say that $\mathbb{X}$ has *coproducts at level l* when every set of parallel l-types has a coproduct. When $\mathbb{X}$ has coproducts at every level, we simply say that $\mathbb{X}$ *has coproducts.*

Remark 2.9.3.2. Let $S \subseteq [\mathbf{d}]$. There are a number of ways of defining the 2-category $\mathrm{GlobMult}_{\amalg}^{S}$ of $\mathbf{d}$-globular multicategories with coproducts at level l for each $l \in S$. Coproducts are unique up to unique isomorphism, and so it is natural to consider the objects of $\mathrm{GlobMult}_{\amalg}^{S}$ to be globular multicategories such that the required coproducts exist, together with homomorphisms sending each coproduct to *some* coproduct. On the other hand, we could consider globular multicategories together with a *choice* of coproduct, and homomorphisms which preserve these choices, either up to isomorphism, or strictly. Finally, it can be useful to require that these choices of coproducts satisfy certain natural properties. For example, we might require that the coproduct of a one element set $\{M\}$ is exactly M. The advantage of some of these definitions over others is that certain properties which, a priori, hold up to isomorphism may in fact hold on the nose. Fortunately, all these ways of defining the 2-category $\mathrm{GlobMult}_{\amalg}^{S}$ are equivalent, and for our purposes, will not need to worry about the precise choice of definition.

Remark 2.9.3.3. A different notion of coproduct for monoidal globular categories is studied in [6][§5].

Example 2.9.3.4. Suppose that $\mathbb{X}$ is representable. Then $\mathbb{X}$ has coproducts at level l if and only if the following conditions hold:

- Each set of parallel l-types $\{M_i\}_{i \in I}$ has a coproduct $\coprod_{i \in I} M_i$ in the category $\mathrm{Type}_l\,\mathbb{X}$, and these coproducts can be chosen such that whenever $\iota : M_j \to \coprod_{i \in I} M_i$ is a canonical projection, we have that $\iota : \mathrm{id}_A \;\not\to\; \mathrm{id}_B$.
- For each $k \in [d]$ such that $1 \leq k \leq \mathbf{d} - l$, the unit functor $\mathbf{Z}^k : \mathrm{Type}_l\,\mathbb{X} \to \mathrm{Type}_{l+k}\,\mathbb{X}$ preserves these coproducts.
- Whenever $k < l \leq n$, composition of n-types along k-types, $- \otimes_k -$, commutes with these coproducts on both sides.

In particular, $\mathbb{X} = \mathrm{Span}_{\mathbf{d}}(\mathcal{C})$ has coproducts if and only if $\mathcal{C}$ has small coproducts and these coproducts are stable under pullback.

Example 2.9.3.5. The globular multicategory $\mathrm{Fam}_{\mathbf{d}}^{[\mathbf{d}]}\,\mathbb{X}$ has coproducts. We define the coproduct of a collection $\{M_r : A \;\not\to\; B\}_{r \in R}$ of l-types in $\mathrm{Fam}_{\mathbf{d}}^{[\mathbf{d}]}\,\mathbb{X}$ by setting

$$\mathbf{I}_{\coprod_r M_r}(i, j) = \coprod_{r \in R} \mathbf{I}_{M_r(i,j)}.$$

Hence, an element of $\mathbf{I}_{\coprod\limits_{r} M_r}(i,j)$ amounts to a pair (r,k), where $r \in R$, and $k \in \mathbf{I}_{M_r(i,j)}$. Hence, we define

$$\left(\coprod_{r\in R} M_r\right)(r,k) = M_r(k).$$

The inclusions are now induced by the universal property of coproducts in Set. This argument also shows that $\operatorname{Fam}^S_{\mathbf{d}}\mathbb{X}$ has coproducts at level l for each $l \in S$. This assignment extends to a strict 2-functor $\operatorname{Fam}^S_{\mathbf{d}} : \mathbf{d}\text{ - GlobMult} \to \mathbf{d}\text{ - GlobMult}^S_{\amalg}$.

We can now state the universal property of families constructions.

Theorem 2.9.3.6. *Let* $S \subseteq [\mathbf{d}]$. *The families construction,* $\operatorname{Fam}^S_{\mathbf{d}} : \text{GlobMult} \to \text{GlobMult}^S_{\amalg}$, *is the (weak) left adjoint of the* 2*-functor* $U^S_{\amalg} : \text{GlobMult}^S_{\amalg} \to \text{GlobMult}$ *that forgets coproducts at level* l *for* $l \in S$.

$$\begin{tikzcd}
\mathbf{d}\text{ - GlobMult} \arrow[r, bend left, "\operatorname{Fam}^S_{\mathbf{d}}"] \arrow[r, phantom, "\perp"] & \mathbf{d}\text{ - GlobMult}^S_{\amalg} \arrow[l, bend left, "U^S_{\amalg}"]
\end{tikzcd}$$

Proof. First suppose that $\mathbb{X}$ is a $\mathbf{d}$-globular multicategory. For each n-type M in $\mathbb{X}$, let $\star_M : \{\star\} \to \operatorname{Type}\mathbb{X}$ be the constant function such that $\star_M(\star) = M$. Then we define the unit $\eta^S_{\amalg}(\mathbb{X}) : \mathbb{X} \to U^S_{\amalg}\operatorname{Fam}^S_{\mathbf{d}}\mathbb{X}$ to be the homomorphism sending each n-type $M \in \mathbb{X}$ to the following one-element family:

$$\mathbf{I}_{\eta^S_{\amalg}(\mathbb{X})(M)}(\star,\star) = \{\star\}, \qquad \eta^S_{\amalg}(\mathbb{X})(M) = \star_M.$$

Suppose that $\mathbb{Y}$ has coproducts at level l for $l \in S$. Then we define the counit $\epsilon^S_{\amalg}(\mathbb{Y}) : \operatorname{Fam}^S_{\mathbf{d}} U^S_{\amalg}\mathbb{Y} \to \mathbb{Y}$ to be the homomorphism such that

$$\epsilon^S_{\amalg}(\mathbb{Y})(M) = \coprod_{k\in\mathbf{I}_M} M(k) : \coprod_{i\in\mathbf{I}_{sM}} sM(i) \twoheadrightarrow \coprod_{j\in\mathbf{I}_{tM}} tM(j)$$

The triangle identities hold up to isomorphism since, for each n-type M in $U^S_{\amalg}\mathbb{Y}$, we have that

$$(U^S_{\amalg}\epsilon^S_{\amalg} \circ \eta^S_{\amalg}U^S_{\amalg})(\mathbb{Y})(M) = \coprod_{k\in\{\star\}} \star_M(\star) = \coprod_{k\in\{\star\}} M \cong M$$

and, for each n-type M in $\operatorname{Fam}^S_{\mathbf{d}}\mathbb{X}$, we have that

$$(\epsilon^S_{\amalg}\operatorname{Fam}^S_{\mathbf{d}} \circ \operatorname{Fam}^S_{\mathbf{d}}\eta^S_{\amalg})(\mathbb{X})(M) = \coprod_{k\in\mathbf{I}_M} (\{\star\}, \star_{M(k)})$$

and so

$$\mathbf{I}_{(\epsilon^S_{\amalg}\,\mathrm{Fam}^S_{\mathbf{d}} \circ \mathrm{Fam}^S_{\mathbf{d}}\,\eta^S_{\amalg})(\mathbb{X})(M)} = \coprod_{k\in\mathbf{I}_M}\{\star\} \cong \mathbf{I}_M$$
$$(\epsilon^S_{\amalg}\,\mathrm{Fam}^S_{\mathbf{d}} \circ \mathrm{Fam}^S_{\mathbf{d}}\,\eta^S_{\amalg})(\mathbb{X})(M)(k,\star) = \star_{M(k)}(\star) = M$$

□

Remark 2.9.3.7. In particular, when $S = [\mathbf{d}]$, it follows that the total families construction is left adjoint to the functor which forgets coproducts at all levels.

Example 2.9.3.8. Combining Theorem 2.9.3.6 with Example 2.9.1.2, we find that SpanSet is the free coproduct completion of $\mathbb{1}$.

Suppose that $S \subseteq [d]$, and that $T \subseteq [d] \setminus S$. Then the same proof shows that we have a 2-adjunction:

$$\begin{tikzcd}
\mathbf{d}\text{ - }\mathrm{GlobMult}^S_{\amalg} \arrow[r, bend left, "\mathrm{Fam}^T"] \arrow[r, phantom, "\perp"] & \mathbf{d}\text{ - }\mathrm{GlobMult}^{S\cup T}_{\amalg} \arrow[l, bend left, "U^T_{\amalg}"]
\end{tikzcd}$$

As a consequence of these universal properties, in order to freely add coproducts at certain levels, we can iteratively apply families constructions at those levels in any order. For example. when n is finite this allows us to re-obtain the total families construction, $\mathrm{Fam}^{[n]}_n$, up to natural equivalence, as the composite:

$$\mathrm{GlobMult} \xrightarrow{\mathrm{Fam}^{\{0\}}_n} \mathrm{GlobMult}^0_{\amalg} \xrightarrow{\mathrm{Fam}^{\{1\}}_n} \dots \xrightarrow{\mathrm{Fam}^{\{n\}}_n} \mathrm{GlobMult}^{\{0,\dots,n\}}_{\amalg}$$

More generally, even when $S = \{l_0, l_1, \dots\}$ is infinite, Fam^S is naturally equivalent to the 2-colimit of the following diagram:

$$\mathrm{GlobMult} \xrightarrow{\mathrm{Fam}^{\{l_0\}}} \dots \xrightarrow{\mathrm{Fam}^{\{l_n\}}} \mathrm{GlobMult}^{\{l_0,\dots,l_n\}}_{\amalg} \xrightarrow{\mathrm{Fam}^{\{l_{n+1}\}}} \dots$$

In particular, When $S = [\omega]$, the total families construction, $\mathrm{Fam}^{[\omega]}_{\omega}$, is the 2-colimit of the following diagram:

$$\omega\text{ - }\mathrm{GlobMult} \xrightarrow{\mathrm{Fam}^{\{0\}}} \dots \xrightarrow{\mathrm{Fam}^{\{n\}}} \mathrm{GlobMult}^{\{0,\dots,n\}}_{\amalg} \longrightarrow \dots$$

Remark 2.9.3.9. Suppose that $\mathbb{X}$ is a globular multicategory with coproducts. Consider the following diagram in GlobMult:

$$\dots \xrightarrow{\epsilon^1_{\amalg}} \mathrm{Fam}^{[1]}\,U^{[1]}_{\amalg}\mathbb{X} \xrightarrow{\epsilon^0_{\amalg}} \mathrm{Fam}^{[0]}\,U^{[0]}_{\amalg}\mathbb{X} \xrightarrow{\epsilon^{-1}_{\amalg}} \mathbb{X}$$

Here, $\epsilon^i_{\amalg}$ is the canonical arrow, taking coproducts at level i, that is induced by the counit of the adjunction $U^{\{i\}}_{\amalg} \vdash \mathrm{Fam}^{\{i\}}$. Then it is easily seen that $\mathrm{Fam}^{[\omega]}\,\mathbb{X}$ is the limit of this diagram in GlobMult.

Chapter 3

Strict Homomorphism Types

We now describe what it means for an n-globular multicategory to have homomorphism types. A 1-globular multicategory has homomorphism types when each 0-type A comes with a 1-type $\mathcal{H}_A : A \nrightarrow A$, together with a reflexivity term $\mathfrak{r}_A : A \to \mathcal{H}_A, \quad \mathrm{id}_A \nrightarrow \mathrm{id}_A$ satisfying an analogue of the Yoneda lemma. In higher dimensions, each type comes with a whole tower of homomorphism types,

$$A, \mathcal{H}_A, \mathcal{H}_A^2, \mathcal{H}_A^3, \dots,$$

resembling the towers of identity types present in intensional type theory. For each level of this tower, we require a reflexivity term that satisfies an analogue of the Yoneda Lemma.

The functor forgetting homomorphism types has a right adjoint:

$$\begin{tikzcd}
\mathrm{GlobMult}_{\mathcal{H}} \arrow[r, bend left, "U_{\mathcal{H}}"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult} \arrow[l, bend left, "\mathrm{Mod}"]
\end{tikzcd}$$

A one-dimensional analogue of this result is already known: Crutwell and Shulman [17] have shown that the monoids and modules construction on virtual double categories (see [17, 30, 31]), first defined by Leinster, has a universal property of this form. We call our higher-dimensional right adjoint the *strict higher modules construction.* In the spirit of [20], we describe how the strict higher modules construction can be obtained by applying a generalisation of the 1-dimensional monoids and modules construction at each level (dimension) of a globular multicategory.

Many fundamental objects in category theory are the result of applying the monoids and modules construction [17]. Perhaps the most well-known result in this direction is that a monoid in the bicategory of spans of sets is precisely a category.

Analogously, the data obtained using the strict higher modules construction turn out to be fundamental objects in strict higher category theory. We describe how the globular multicategory of strict higher modules in SpanSet is equivalent to the collection of strict ω-categories, strict profunctors between strict ω-categories, and strict higher transformations between these objects. We also show how strict higher modules constructions are closely related to iterated enrichment.

3.1 Degeneracies

Suppose that $x : A$ is a variable in a context Γ. Then, there is a context, $\Gamma \oplus_x \mathcal{H}_A$, obtained from Γ by *adding a homomorphism type at* x; when studying homomorphism types, we will frequently encounter contexts of this form. Similarly, we will often speak of substitutions with a term (typically a reflexivity term) added at some variable. Consequently, it will be useful to have an explicit description of these contexts and substitutions. The types and terms added in this way are always *degenerate,* in the sense that each such type (or term) has the same source and target type (or term). Hence, we will first describe a process which *adds a degeneracy* to a labelled pasting diagram, and then specialise this discussion to understand adding a degeneracy to contexts and substitutions, which are, by definition, certain labelled pasting diagrams.

3.1.1 Adding degeneracies to pasting diagrams

Suppose that $0 \leq k < n$. Let π be an n-pasting diagram, and let x be a k-cell of π. In this section, we describe an operation that "adds a $(k+1)$-cell at x". Let $\partial\mathbf{D}^{k+1}$ be the subobject of the representable $\mathbf{D}^{k+1}$ such that

$$x \in \partial\mathbf{D}^{k+1}(l) \iff l < k+1.$$

Thus, $\partial\mathbf{D}^{k+1}$ is generated by a pair of parallel k-cells. For example, the following diagram depicts $\partial\mathbf{D}^{2}$:

$$\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet
\end{tikzcd}$$

Let $\delta : \partial\mathbf{D}^{k+1} \to \mathbf{D}^{k+1}$ be the canonical subobject inclusion. Let $\nabla : \partial\mathbf{D}^{k+1} \to \mathbf{D}^{k}$ be the map which identifies the two k-cells of $\partial\mathbf{D}^{k+1}$. We define the globular set $\pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}}$ to be the following pushout in $\mathbb{G}$-Set:

$$\begin{tikzcd}
\partial\mathbf{D}^{k+1} \arrow[r, "\nabla"] \arrow[d, "\delta"'] & \mathbf{D}^{k} \arrow[r, "x"] & \pi \arrow[d] \\
\mathbf{D}^{k+1} \arrow[rr, "\bar{\mathcal{H}}_x"'] & & \pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}}
\end{tikzcd}$$

Thus, whenever X is a globular set, to give a map $\pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}} \to X$ is to give a map $f : \pi \to X$, together with a $(k+1)$-cell $\bar{\mathcal{H}}_x$ such that $s\bar{\mathcal{H}}_x = t\bar{\mathcal{H}}_x = fx$. For example, when π is the following 2-pasting diagram,

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, "y"] & C \end{tikzcd} \tag{3.1.1.a}$$

and x and y are as labelled in this diagram, then $\pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}}$ is the following globular set:

$$\begin{tikzcd} x \arrow[loop left, "\bar{\mathcal{H}}_x"] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, "y"] & C \end{tikzcd} \tag{3.1.1.b}$$

, and $\pi \mathbin{\bar{\oplus}_y} \bar{\mathcal{H}}$ is the following globular set:

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, "y"] \arrow[r, phantom, "\circlearrowright\,\bar{\mathcal{H}}_y"', shift right=3] & C \end{tikzcd} \tag{3.1.1.c}$$

As these examples illustrate, for any choice of π and x, the globular set $\pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}}$ is not a pasting diagram: the added cell $\bar{\mathcal{H}}_x$ has the same source and target, namely x, and no cell in a pasting diagram can have this property. Nonetheless, we can define a pasting diagram that approximates $\pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}}$.

Definition 3.1.1.1. Suppose that $0 \leq k < n$. Suppose that $\pi = (\pi_1, \ldots, \pi_l)$ is an n-pasting diagram, and that x is a k-cell of π. We define an n-pasting diagram $\pi \oplus_x \mathcal{H}$ together with a $(k+1)$-cell $\mathcal{H}_x$ in $\pi \oplus_x \mathcal{H}$ by induction on k.

First suppose that $k = 0$. Then $x = (i)$ for some $0 \leq i \leq l$, and we define

$$\begin{aligned} \pi \oplus_x \mathcal{H} &= (\pi_1, \ldots, \pi_i, \star, \pi_{i+1}, \ldots, \pi_l) \\ &= (\pi_1, \ldots, \pi_i) \odot_0 \mathbf{D}^1 \odot_0 (\pi_{i+1}, \ldots, \pi_l) \end{aligned}$$

We define $\mathcal{H}_x \in (\pi \oplus_x \mathcal{H})(1)$ to be the unique 1-cell of the summand $\mathbf{D}^1$.

Now suppose that $k > 0$, and that $x = (i, x')$ for some $1 \leq i \leq l$ and some $x' \in \pi_i(k-1)$. By induction, we have already defined $\pi_i \oplus_{x'} \mathcal{H}$. Hence, we define

$$\begin{aligned} \pi \oplus_x \mathcal{H} &= (\pi_1, \ldots, \pi_{i-1}, \pi_i \oplus_{x'} \mathbf{D}^k, \pi_{i+1}, \ldots, \pi_l) \\ &= (\pi_1, \ldots, \pi_{i-1}) \odot_0 \Sigma(\pi_i \oplus_{x'} \mathcal{H}) \odot_0 (\pi_{i+1}, \ldots, \pi_l) \end{aligned}$$

By induction, we have defined a k-cell $\mathcal{H}_{x'}$ in $\pi_i \oplus_{x'} \mathcal{H}$. Hence, we define $\mathcal{H}_x$ to be the corresponding $(k+1)$-cell in the summand $\Sigma(\pi_i \oplus_{x'} \mathcal{H})(k+1)$.

Example 3.1.1.2. Suppose again that π is the 2-pasting diagram (3.1.1.a)

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, "y"] & C \end{tikzcd}$$

and x and y are as labelled in this diagram. Then, $\pi \oplus_x \mathcal{H}$ is the following pasting diagram:

$$\begin{tikzcd} x_0 \arrow[r, "\mathcal{H}_x"] & x_1 \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, "y"] & C \end{tikzcd} \tag{3.1.1.d}$$

and $\pi \oplus_y \mathcal{H}$ is the following pasting diagram:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & B \arrow[r, bend left, "y_0"] \arrow[r, bend right, "y_1"'] \arrow[r, phantom, "\Downarrow\mathcal{H}_y"] & C \end{tikzcd} \tag{3.1.1.e}$$

In the preceding examples and throughout this thesis, we adopt the convention that $s\mathcal{H}_x = x_0$ and $t\mathcal{H}_x = x_1$. This illustrates the key difference between $\pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}}$ and $\pi \oplus_x \mathcal{H}$: the former adds a *loop* $\bar{\mathcal{H}}_x$ at x such that $s\bar{\mathcal{H}}_x = t\bar{\mathcal{H}}_x = x$, while the latter replaces x by a $(k+1)$-cell whose source and target are distinct k-cells. The following result now formalises this observation and describes how $\pi \oplus_x \mathcal{H}$ approximates $\pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}}$.

Proposition 3.1.1.3. *There exists a map of globular sets* $\mathbf{Q}^{\pi}_x : \pi \oplus_x \mathcal{H} \to \pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}}$ *such that, for any globular set* X*, to give a map*

$$f : \pi \oplus_x \mathcal{H} \longrightarrow X$$

such that $sf(\mathcal{H}_x) = tf(\mathcal{H}_x)$ *is to give a map*

$$\bar{f} : \pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}} \longrightarrow X$$

such that $f = \bar{f} \circ Q^{\pi}_x$.

Proof. We proceed by induction on k. First, suppose that $k = 0$. Then we define $\mathbf{Q}^{\pi}_x$ to be the map induced by including $(\pi_1, \ldots, \pi_i)$ and $(\pi_{i+1}, \ldots, \pi_l)$ into $\pi = (\pi_1, \ldots, \pi_l)$, and by sending $\mathcal{H}_x$ to $\bar{\mathcal{H}}_x$. Now suppose that $k > 0$. Then $x = (i, x')$. We define $\mathbf{Q}^{\pi}_x$ to be the map induced by including $(\pi_1, \ldots, \pi_i)$ and $(\pi_{i+1}, \ldots, \pi_l)$ into $\pi = (\pi_1, \ldots, \pi_l)$, and by defining the component of $\mathbf{Q}^{\pi}_x$ at $\Sigma(\pi_i \oplus_{x'} \mathcal{H})$ to be $\Sigma\mathbf{Q}^{\pi_i}_{x'}$. The required property of $\mathbf{Q}^{\pi}_x$ is now easily verified. ☐

Example 3.1.1.4. Suppose once more that π is the 2-pasting diagram (3.1.1.a). The map $\mathbf{Q}^{\pi}_{x} : \pi \oplus_x \mathcal{H} \to \pi \bar{\oplus}_x \bar{\mathcal{H}}$ is the quotient map from (3.1.1.d) to (3.1.1.b) sending $x_0 \mapsto x$, $x_1 \mapsto x$ and $\mathcal{H}_x \to \bar{\mathcal{H}}_x$ and fixing all other labels. The map $\mathbf{Q}^{\pi}_{y}$ is the quotient map from (3.1.1.e) to (3.1.1.c) sending $y_0 \mapsto y$, $y_1 \mapsto y$ and $\mathcal{H}_y \to \bar{\mathcal{H}}_y$ and fixing all other labels.

Proposition 3.1.1.5. *Suppose that π is an n-pasting diagram. Suppose that x is a k-variable of π such that $k < n$. If $x \notin s_k\pi$. then we have the following alternative description of $\pi \oplus_x \mathcal{H}$:*

$$\pi \oplus_x \mathcal{H} = \bigodot_{y \in \pi} p^{\pi,x}_{y}$$

where

$$p^{\pi,x}_{y} = \begin{cases} \mathbf{D}^{k+1} \odot_k \mathbf{D}^{\dim y} & \textit{if } \dim y > k \textit{ and } s_k y = x \\ \mathbf{D}^{\dim y} & \textit{otherwise} \end{cases}$$

Similarly, if $x \notin t_k\pi$, then

$$\pi \oplus_x \mathcal{H} = \bigodot_{y \in \pi} q^{\pi,x}_{y}$$

where

$$q^{\pi,x}_{y} = \begin{cases} \mathbf{D}^{\dim y} \odot_k \mathbf{D}^{k+1} & \textit{if } \dim y > k \textit{ and } t_k y = x \\ \mathbf{D}^{\dim y} & \textit{otherwise} \end{cases}$$

Proof. Suppose that ρ is a k-trivial pasting diagram. Then, we have that

$$\mathbf{D}^{k+1} = \Sigma^{k+1}\mathbf{D}^0 = \Sigma^{k+1} \bigodot_{y \in \rho} \mathbf{D}^0 = \bigodot_{y \in \Sigma^{k+1}\rho} d^{k+1}_{y}$$

where

$$d^{k+1}_{y} = \begin{cases} \Sigma^{k+1}\mathbf{D}^0 & \text{if } \dim y > k \\ \mathbf{D}^{\dim y} & \text{otherwise} \end{cases}$$
$$= \begin{cases} \mathbf{D}^{k+1} & \text{if } \dim y > k \\ \mathbf{D}^{\dim y} & \text{otherwise} \end{cases}$$

Hence,

$$\rho \odot_k \mathbf{D}^{k+1} = \bigodot_{y \in \rho} (\mathbf{D}^{\dim y} \odot_k d^{k+1}_{y})$$

and

$$\mathbf{D}^{k+1} \odot_k \rho = \bigodot_{y \in \rho} (d^{k+1}_{y} \odot_k \mathbf{D}^{\dim y}).$$

We will now prove the first identity of the proposition. The second follows by a similar argument. Suppose that $\pi = (\pi_1, \ldots, \pi_l)$. We proceed by induction on k. First, suppose that $k = 0$. Then, $x = (i)$ for some $0 \leq i \leq l$. For each cell y in π such that $\dim y > k$, we have that $s_k y = x \iff y \in \pi_i$. Combining this with the fact that (π_i) is 0-trivial, we obtain

$$\begin{aligned}
\pi \oplus_x \mathcal{H} &= (\pi_1, \ldots, \pi_i) \odot_0 \mathbf{D}^1 \odot_0 (\pi_{i+1}, \ldots, \pi_l) \\
&= (\pi_1, \ldots, \pi_{i-1}) \odot_0 (\pi_i \odot_0 \mathbf{D}^1) \odot_0 (\pi_{i+1}, \ldots, \pi_l) \\
&= \bigodot_{y \in (\pi_1, \ldots, \pi_{i-1})} (\mathbf{D}^{\dim y}) \odot_0 \bigodot_{y \in (\pi_i)} (\mathbf{D}^{\dim y} \odot_0 d_y^1) \odot_0 \bigodot_{y \in (\pi_{i+1}, \ldots, \pi_l)} (\mathbf{D}^{\dim y}) \\
&= \bigodot_{y \in \pi} p_y^{\pi, x}
\end{aligned}$$

Now suppose that $k > 0$, and $x = (i, x')$. By the inductive hypothesis, we have that $\pi_i \oplus_{x'} \mathcal{H}_{x'} = \bigodot_{y \in \pi_i} p_y^{\pi_i, x'}$. Furthermore, it is easily seen that

$$\Sigma p_y^{\pi, x'} = p_y^{\Sigma \pi, x'}.$$

Hence,

$$\Sigma(\pi_i \oplus_{x'} \mathcal{H}_{x'}) = \bigodot_{y \in \Sigma \pi_i} p_y^{\Sigma \pi_i, x'},$$

However, by the inductive hypothesis, we have that

$$\begin{aligned}
\pi \oplus_x \mathcal{H} &= (\pi_1, \ldots, \pi_{i-1}, \pi_i \oplus_{x'} \mathcal{H}_{x'}, \pi_{i+1}, \ldots, \pi_l) \\
&= (\pi_1, \ldots, \pi_{i-1}) \odot_0 \Sigma(\pi_i \oplus_{x'} \mathcal{H}) \odot_0 (\pi_{i+1}, \ldots, \pi_l) \\
&= \bigodot_{y \in (\pi_1, \ldots, \pi_{i-1})} (\mathbf{D}^{\dim y}) \odot_0 \bigodot_{y \in (\pi_i)} (p_y^{\pi, x}) \odot_0 \bigodot_{y \in (\pi_{i+1}, \ldots, \pi_l)} (\mathbf{D}^{\dim y}) \\
&= \bigodot_{y \in \pi} p_y^{\pi, x}.
\end{aligned}$$

□

The following two propositions follow straightforwardly from the definition of $\pi \oplus_x \mathcal{H}_A$.

Proposition 3.1.1.6. *Suppose that $0 \le k < n$. Suppose that π is an n-pasting diagram, and that x is a k-cell of π. We have that*

$$\pi \oplus_x \mathcal{H} \oplus_{x_0} \mathcal{H} = \pi \oplus_x \mathcal{H} \oplus_{x_1} \mathcal{H}.$$

Proposition 3.1.1.7. *Suppose that π is an n-pasting diagram. Suppose that $x \neq y$ are distinct cells of π such that $\dim x < n$ and $\dim y < n$. Then,*

$$\pi \oplus_x \mathcal{H} \oplus_y \mathcal{H} = \pi \oplus_y \mathcal{H} \oplus_x \mathcal{H}.$$

These propositions allow us to understand repeated applications of $- \oplus -$. Suppose that $S\{x^1, \dots, x^l\} \subseteq \pi(k)$ is a set of k-cells. Then, repeatedly applying Definition 3.1.1.1, we obtain a pasting diagram

$$\pi \oplus_S \mathcal{H} = \pi \oplus_{x^1} \mathcal{H} \oplus_{x^2} \mathcal{H} \oplus_{x^3} \cdots \oplus_{x^l} \mathcal{H}$$

Furthermore, this pasting diagram does not depend on the order of the x_i. The following alternative description of $\pi \oplus \mathcal{H}_S$ generalises Proposition 3.1.1.5, and follows by a similar argument.

Proposition 3.1.1.8. *Suppose that π is an n-pasting diagram. Suppose that x is a k-variable of π such that $k < n$. If $x \cap s_k\pi(k) = \emptyset$. Then, we have the following alternative description of $\pi \oplus_x \mathcal{H}$:*

$$\pi \oplus_S \mathcal{H} = \bigodot_{y \in \pi} p_y^{\pi,x}$$

where

$$p_y^{\pi,x} = \begin{cases} \mathbf{D}^{k+1} \odot_k \mathbf{D}^{\dim y} & \textit{if } \dim y > k \textit{ and } s_k y \in S \\ \mathbf{D}^{\dim y} & \textit{otherwise} \end{cases}$$

Similarly, if $x \cap t_k\pi(l) = \emptyset$, then

$$\pi \oplus_S \mathcal{H} = \bigodot_{y \in \pi} q_y^{\pi,x}$$

where

$$q_y^{\pi,x} = \begin{cases} \mathbf{D}^{\dim y} \odot_k \mathbf{D}^{k+1} & \textit{if } \dim y > k \textit{ and } t_k y \in S \\ \mathbf{D}^{\dim y} & \textit{otherwise} \end{cases}$$

3.1.2 Adding degeneracies to contexts and substitutions

Suppose that $\mathbb{X}$ is a globular multigraph, that Γ is a π-shaped n-context in $\mathbb{X}$, and that $x : A$ is a k-variable in Γ. Then, whenever $H : A \nrightarrow A$ is an $(k+1)$-type in $\mathbb{X}$, we define the $(\pi \oplus_x \mathcal{H})$-shaped n-context, $\Gamma \oplus_x H$, to be the map $\pi \oplus_x \mathcal{H} \to \operatorname{Type} X$ induced by Definition 3.1.1.1 and the following dotted arrow:

$$\begin{tikzcd}
\partial\mathbf{D}^{k+1} \arrow[r, "\nabla"] \arrow[d, "\delta"'] & \mathbf{D}^k \arrow[r, "x"] & \pi \arrow[d] \arrow[rdd, bend left, "\Gamma"] & \\
\mathbf{D}^{k+1} \arrow[rr] \arrow[rrrd, bend right, "H"'] & & \pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}} \arrow[rd, dotted] & \\
& & & \operatorname{Type} \mathbb{X}
\end{tikzcd}$$

Suppose that $f : \Gamma \to \Delta$ is a π-shaped substitution, and that y is a k-variable in Δ. Suppose that $h : f_y \twoheadrightarrow f_y$ is a $(k+1)$-term in $\mathbb{X}$. Then, similarly, there is a canonical $(\pi \oplus_x \mathcal{H})$-shaped substitution $f \oplus_y h$ induced by Definition 3.1.1.1 and the following dotted arrow:

$$\begin{tikzcd}
\partial\mathbf{D}^{k+1} \arrow[r, "\nabla"] \arrow[d, "\delta"'] & \mathbf{D}^k \arrow[r, "y"] & \pi \arrow[d] \arrow[rdd, bend left, "f"] & \\
\mathbf{D}^{k+1} \arrow[rr] \arrow[rrrd, bend right, "h"'] & & \pi \mathbin{\bar{\oplus}_x} \bar{\mathcal{H}} \arrow[rd, dotted] & \\
 & & & \mathrm{Term}\,\mathbb{X}
\end{tikzcd}$$

More generally, suppose that $S = \{x_1 : A_1, \dots, x_m : A_m\} \subseteq \Gamma(k)$ is a set of k-variables. Suppose that for each i, we have a type $\mathcal{H}_{A_i} : A_i \twoheadrightarrow A_i$. Then there is a canonical $\pi \oplus_S \mathcal{H}$-shaped context $\Gamma \oplus_S \mathcal{H}$ defined by

$$\Gamma \oplus_S H \; = \Gamma \oplus_{x_1} \mathcal{H}_{A_1} \oplus_{x_2} \cdots \oplus_{x_m} \mathcal{H}_{A_m}$$

This context does not depend on the order of the x_i.

Remark 3.1.2.1. By Proposition 3.1.1.5, if $S \cap s_k\pi(k) = \emptyset$, we have that

$$\Gamma \oplus_S \mathcal{H} = \bigodot_{y:B\in\Gamma} P_y^{\pi,x}$$

where

$$P_y^{\pi,x} = \bigodot_{y:B\in\Gamma} \begin{cases} \mathcal{H}_{A_i} \odot_k & \text{if } \dim y > k \text{ and } s_k y = x_i \\ B & \text{otherwise} \end{cases}$$

Similarly, if $S \cap t_k\pi(k) = \emptyset$, then for each variable $y : B$ in Γ, we have that

$$\Gamma \oplus_S \mathcal{H} = \bigodot_{y:B\in\Gamma} Q_y^{\pi,x}$$

where

$$Q_y^{\pi,x} = \bigodot_{y:B\in\Gamma} \begin{cases} B \odot_k \mathcal{H}_{A_i} & \text{if } \dim y > k \text{ and } t_k y = x_i \\ B & \text{otherwise} \end{cases}$$

A similar statement holds for terms and substitutions.

3.2 Definition

Now that we have defined what it means to add degenerate types to a context, we can describe what it means for a globular multicategory to have homomorphism types. Firstly, we require that each n-type A be equipped a *homomorphism* $(n+1)$*-type* $\mathcal{H}_A$ and a *reflexivity term* $\mathfrak{r}_A : A \to \mathcal{H}_A$. Secondly, composition with reflexivity terms should give a bijection between terms which removes homomorphism types from source contexts.

Definition 3.2.0.1. A *reflexive globular multigraph* consists of a globular multigraph $\mathbb{X}$ together with, for each n-type A, a *homomorphism* $(n+1)$*-type*

$$\mathcal{H}_A : A \nrightarrow A$$

and a *reflexivity* $(n+1)$*-term*

$$\mathfrak{r}_A : A \longrightarrow \mathcal{H}_A, \qquad \mathrm{id}_A \nrightarrow \mathrm{id}_A\,.$$

A *reflexive globular multicategory* is a globular multicategory together with a choice of reflexive structure for its underlying globular multigraph.

Definition 3.2.0.2. Suppose that $\mathbb{X}$ is a reflexive globular multigraph. Let $0 \leq k < n$. Suppose that Γ is a π-shaped n-context in $\mathbb{X}$, and that $x : A$ is a k-variable in Γ. It follows that we have a canonical $(\pi \oplus_x \mathcal{H}_A)$-shaped context $\Gamma \oplus_x \mathcal{H}_A$. We define the $(\pi \oplus_x \mathcal{H}_A)$-shaped *reflexivity substitution*

$$\mathfrak{r}^{\Gamma}_x : \Gamma \to \Gamma \oplus_x \mathcal{H}_A$$

by $\mathfrak{r}^{\Gamma}_x = \mathrm{id}_\Gamma \oplus_x \mathfrak{r}_A$. When Γ is clear from the context, we will simply denote this substitution by $\mathfrak{r}_x$. When $k = n-1$, we have that

$$\Gamma \oplus_x \mathcal{H}_A : s\Gamma \nrightarrow t\Gamma, \qquad \mathfrak{r}^{\Gamma}_x : \mathrm{id}_{s\Gamma} \nrightarrow \mathrm{id}_{t\Gamma},$$

and, when $k < n-1$, we have that

$$\Gamma \oplus_x \mathcal{H}_A : s\Gamma \oplus_x \mathcal{H}_A \nrightarrow t\Gamma \oplus_x \mathcal{H}_A, \qquad \mathfrak{r}^{\Gamma}_x : \mathfrak{r}^{s\Gamma}_x \nrightarrow \mathfrak{r}^{t\Gamma}_x\,.$$

More generally, suppose that $S = \{x_1 : A_1, \ldots, x_m : A_m\}$ is a set of k-variables in Γ, for some $k < n$. Then, we define a context $\Gamma \oplus_S \mathcal{H}$, and a substitution $\mathfrak{r}^{\Gamma}_S : \Gamma \to \Gamma \oplus_S \mathcal{H}$ by

$$\Gamma \oplus_S \mathcal{H} = \Gamma \oplus_{x_1} \mathcal{H}_{A_1} \oplus_{x_2} \cdots \oplus_{x_m} \mathcal{H}_{A_m}, \qquad \mathfrak{r}^{\Gamma}_S = \mathrm{id}_\Gamma \oplus_{x_1} \mathfrak{r}_{A_1} \oplus_{x_2} \cdots \oplus_{x_m} \mathfrak{r}_{A_M}\,.$$

This definition does not depend on the order of the x_i.

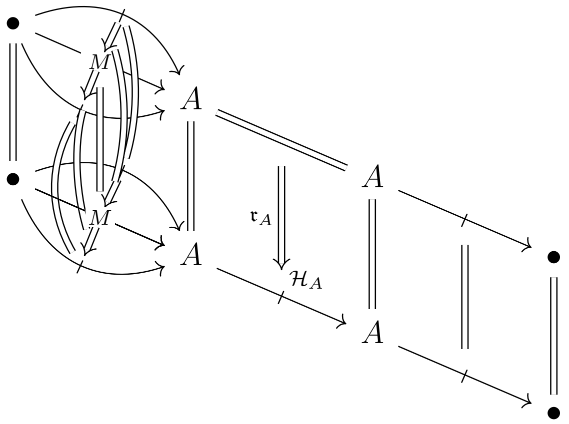
Example 3.2.0.3. Suppose that Γ is the following partially labelled 2-context

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, "M" description, ""{name=C}] \arrow[r, bend right=60, ""{name=D, above}] & A \arrow[r, "\shortmid" marking] & \bullet
\arrow[Rightarrow, from=U, to=C] \arrow[Rightarrow, from=C, to=D]
\end{tikzcd}$$

Suppose that $x : A$ is the 0-variable whose type, A, is labelled in this diagram. Suppose that $y : M$ is the 1-variable whose type, M, is labelled in this diagram. Then $\Gamma \oplus_x \mathcal{H}_A$ is the following context:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, "M" description, ""{name=C}] \arrow[r, bend right=60, ""{name=D, above}] & A \arrow[r, "\mathcal{H}_A", "\shortmid" marking] & A \arrow[r] & \bullet
\arrow[Rightarrow, from=U, to=C] \arrow[Rightarrow, from=C, to=D]
\end{tikzcd}$$

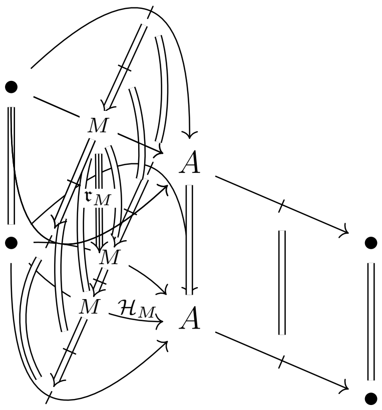
and $\mathfrak{r}_x^\Gamma$ is the following substitution:



On the other hand, we have that $\Gamma \oplus_y \mathcal{H}_M$ is the following context:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=80, ""{name=U, below}] \arrow[r, bend left=20, "M" description, ""{name=C1}] \arrow[r, bend right=20, "M" description, ""{name=C2}] \arrow[r, bend right=80, ""{name=D, above}] & A \arrow[r, "\shortmid" marking] & \bullet
\arrow[Rightarrow, from=U, to=C1] \arrow[Rightarrow, from=C1, to=C2, "\mathcal{H}_M"] \arrow[Rightarrow, from=C2, to=D]
\end{tikzcd}$$

and $\mathfrak{r}_y^\Gamma$ is the following substitution:



Definition 3.2.0.4. Suppose that $\mathbb{X}$ is a reflexive globular multicategory. Suppose that $0 \leq k < n$. Suppose that A is a k-type in $\mathbb{X}$. We say that $\mathbb{X}$ has a *strict homomorphism type at* A when, for each $k < n$, each n-context Γ, each k-variable $x : A$ in Γ, and each n-type M, and each pair of term-wise parallel $(n-1)$-terms $g : s\Gamma \to sM$, $h : t\Gamma \to tM$, the composition map

$$[\Gamma \oplus_x \mathcal{H}_A \longrightarrow M, \quad g \twoheadrightarrow h] \xrightarrow{\mathfrak{r}_x^\Gamma;-} [\Gamma \longrightarrow M, \quad s_{n-1}\mathfrak{r}_x; g \twoheadrightarrow t_{n-1}\mathfrak{r}_x; h]$$

is a bijection with inverse $\mathfrak{J}_x$. We say that $\mathbb{X}$ has *strict homomorphism types* when $\mathbb{X}$ has a homomorphism type at A for each k-type A. We denote the 2-category of globular multicategories with chosen strict homomorphism types, homomorphism type preserving homomorphisms, and transformations between them by $\mathrm{GlobMult}_{\mathcal{H}}$.

Remark 3.2.0.5. Suppose that n is finite. Then, we say that an n-globular mulitcategory $\mathbb{X}$ has *strict homomorphism types* when $\mathbb{X}$ has a homomorphism type at A for each k-type A with $k < n$. We denote the 2-category of globular multicategories with chosen strict homomorphism types, homomorphism type preserving homomorphisms, and transformations between them by n - $\mathrm{GlobMult}_{\mathcal{H}}$.

Remark 3.2.0.6. In the terminology of Definition 2.8.0.6, a globular multicategory has homomorphism types when every reflexivity substitution is strictly representing. Thus, globular multicategories with homomorphism types occupy a middle ground between general globular multicategories and representable globular multicategories.

Remark 3.2.0.7. Suppose that A is a k-type in $\mathbb{X}$ in a globular multicategory. Then, strict homomorphism types of A are unique up to unique isomorphism.

Remark 3.2.0.8. Suppose that $f : \Gamma \to M$ is an n-term in a globular multicategory with homomorphism types. Suppose that $n > l$. Let $S = \{x_1 : A_1, \dots, x_m : A_m\} \subseteq \Gamma(l)$ be a collection of l-variables in Γ. Then there is a unique term

$$\mathfrak{J}_S(f) : \Gamma \oplus_{x_1} \mathcal{H}_{A_1} \oplus_{x_2} \cdots \oplus_{x_n} \mathcal{H}_{A_m} \to M$$

such that $\mathfrak{r}^{\Gamma}_{S} ; \mathfrak{J}_S(f) = f$. We have that

$$\mathfrak{J}_S(f) = \mathfrak{J}_{x_1} \cdots \mathfrak{J}_{x_m}(f)$$

and this definition does not depend on the order of the x_i.

3.3 Examples and Properties

Example 3.3.0.1. Suppose that A is a 0-type in a 1-globular multicategory. Then, a homomorphism type at A consists of a 1-type $\mathcal{H}_A$ together with a 1-term

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathfrak{r}_A"] & A \arrow[d, equal] \\
A \arrow[r, "|" marking, "\mathcal{H}_A"'] & A
\end{tikzcd}$$

such that, for any $m, n \geq 0$, and any sequences 1-types $(M_i : B_{i-1} \nrightarrow B_i)_{0<i\leq m}$, and $(N_i : C_{i-1} \nrightarrow C_i)_{0<i\leq n}$, such that $M_m = A$, and $C_0 = A$, pre-composing with the substitution

$$\begin{tikzcd}
B_0 \arrow[r, "|" marking, "M_1"] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & \dots \arrow[r, "|" marking, "M_m"] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathfrak{r}_A"] & A \arrow[r, "|" marking, "N_1"] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & \dots \arrow[r, "|" marking, "N_n"] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & C_n \arrow[d, equal] \\
B_0 \arrow[r, "|" marking, "M_1"'] & \dots \arrow[r, "|" marking, "M_m"'] & A \arrow[r, "|" marking, "\mathcal{H}_A"'] & A \arrow[r, "|" marking, "N_1"'] & \dots \arrow[r, "|" marking, "N_n"'] & C_n
\end{tikzcd}$$

defines a bijection between terms of the form

$$\begin{tikzcd}
B_0 \arrow[r, "|" marking, "M_1"] \arrow[d] & \cdots \arrow[r, "|" marking, "M_m"] & A \arrow[r, "|" marking, "\mathcal{H}_A"] \arrow[dr, phantom, "\Downarrow"] & A \arrow[r, "|" marking, "N_1"] & \cdots \arrow[r, "|" marking, "N_n"] & C_n \arrow[d] \\
D \arrow[rrrrr, "|" marking, "O"'] & & & & & E
\end{tikzcd}$$

and terms of the form

$$\begin{tikzcd}
B_0 \arrow[r, "|" marking, "M_1"] \arrow[d] & \cdots \arrow[r, "|" marking, "M_m"] & A \arrow[r, "|" marking, "N_1"] \arrow[d, phantom, "\Downarrow"] & \cdots \arrow[r, "|" marking, "N_n"] & C_n \arrow[d] \\
D \arrow[rrrr, "|" marking, "O"'] & & \, & & E
\end{tikzcd}$$

Thus, strict homomorphism type in 1-globular multicategories are precisely the *horizontal units* described by Crutwell and Shulman [17].

Ibid., the monoids and modules construction for virtual double categories is exhibited as the right adjoint of the 2-functor which forgets horizontal units. Many familiar collections of "category-like" objects can be seen as the result of this construction. Hence, any such collection gives rise to a 1-globular multicategory with homomorphism types.

Example 3.3.0.2. Let $\mathcal{C}$ be a monoidal globular category. Then, following Remark 3.2.0.6 the corresponding globular multicategory $U_{\otimes}\mathcal{C}$ has homomorphism types. In order to make this explicit, let A be an n-type in $U_{\otimes}\mathcal{C}$. Then we define $\mathcal{H}_A$ to be the $(n+1)$-type such that

$$\ulcorner \mathcal{H}_A \urcorner = \mathbf{Z}(A).$$

The $(n+1)$-context $[A]$ in $U_{\otimes}\mathcal{C}$ corresponds to the object $\ulcorner [A] \urcorner = \mathbf{Z}(A)$ in $\mathcal{C}$, and so we define the reflexivity term $\mathfrak{r}_A : [A] \to \mathcal{H}_A$ in $U_{\otimes}\mathcal{C}$ to be the term corresponding to the identity arrow

$$\mathrm{id}_{\mathbf{Z}(A)} : \mathbf{Z}(A) \to \mathbf{Z}(A)$$

in $\mathcal{C}$. Now suppose that we have n-context Γ in $U_{\otimes}(\mathcal{C})$, and a k-variable $x : A$ for some $0 \leq k < n$. Then $\ulcorner \mathfrak{r}_x^{\Gamma} \urcorner : \ulcorner \Gamma \urcorner \to \ulcorner \Gamma \oplus_x \mathcal{H}_A \urcorner$ is a coherence law of $\mathcal{C}$ that adds a unit $\mathbf{Z}(A)$. Consequently, $\ulcorner \mathfrak{r}_x^{\Gamma} \urcorner$ has an inverse

$$\ulcorner \Gamma \oplus_x \mathcal{H}_A \urcorner \xrightarrow{u_x^{\Gamma}} \ulcorner \Gamma \urcorner.$$

Whenever $f : \Gamma \to M$ is an n-term in $U_{\otimes}(\mathcal{C})$ we define $\mathfrak{J}_x(f) : \Gamma \oplus_x \mathcal{H}_A \to M$ to be the term in $U_{\otimes}\mathcal{C}$ such that $\ulcorner \mathfrak{J}_x(f) \urcorner$ is the following composite:

$$\ulcorner \Gamma \oplus_x \mathcal{H}_A \urcorner \xrightarrow{u_x^{\Gamma}} \ulcorner \Gamma \urcorner \xrightarrow{\ulcorner f \urcorner} M$$

in $\mathcal{C}$. It follows that $\mathfrak{J}_x$ is the inverse of composition with $\mathfrak{r}_x^{\Gamma}$.

Example 3.3.0.3. For any category with pullbacks, the globular multicategory $\mathrm{Span}(\mathcal{C})$ has strict homomorphism types. For each n-type A, the homomorphism type $\mathcal{H}_A : A \nrightarrow A$ is the trivial span $A \xleftarrow{\mathrm{id}_A} A \xrightarrow{\mathrm{id}_A} A$, and the reflexivity term $\mathfrak{r}_A : A \to \mathcal{H}_A$ is the identity arrow $\mathrm{id}_A : A \to A$.

Example 3.3.0.4. The terminal globular operad $\mathbb{1}$ has strict homomorphism types. We define $\mathcal{H}_n = n+1$, and we define $\mathfrak{r}_n$ to be the unique term $n \to n+1$. For each $f : \pi \to n$, and $x \in \pi$, we define $\mathfrak{J}_x(f)$ to be the unique term such that $\mathfrak{J}_x(f) : \pi \oplus_x \mathcal{H}_A \to n$.

Example 3.3.0.5. Suppose that $\mathbb{P}$ is a contractible globular operad with strict homomorphism types. Then the contraction $! : \mathbb{P} \to \mathbb{1}$ has a strict homomorphism type preserving section $\mathbb{F} : \mathbb{1} \to \mathbb{P}$ defined inductively by

$$\mathbb{F}(\mathfrak{r}_n) = \mathfrak{r}_n, \qquad \mathbb{F}(\mathfrak{J}_x(f)) = \mathfrak{J}_x(\mathbb{F}(f)).$$

See Chapter 5 where we show that $\mathbb{1}$ is freely generated from a 0-type by adding strict homomorphism types. We conjecture that this result implies that algebras of $\mathbb{P}$ are equivalent to strict ω-categories in an appropriate sense.

Proposition 3.3.0.6. *Discrete opfibrations reflect strict homomorphism types.*

Proof. Suppose that $\mathbb{X}$ is a globular multicategory, and that A is an n-type in $\mathbb{X}$ with a strict homomorphism type. Suppose that $\mathbb{F} : \mathbb{Y} \to \mathbb{X}$ is a discrete fibration, and that $\tilde{A}$ is an n-type in $\mathbb{Y}$ such that $\mathbb{F}(\tilde{A}) = A$. We define $\mathfrak{r}_{\tilde{A}} : \tilde{A} \to \mathcal{H}_{\tilde{A}}$ to be the unique term in $\mathbb{Y}$ such that $\mathbb{F}(\mathfrak{r}_{\tilde{A}}) = \mathfrak{r}_A : A \to \mathcal{H}_A$. Suppose that $m > n$, and that Γ is an m-context in $\mathbb{Y}$, and that $x : A$ is a variable in Γ. Suppose that $f : \Gamma \to M$ is an m-term in $\mathbb{Y}$. Then we define $\mathfrak{J}_x(f) : \Gamma \oplus_x \mathcal{H}_A \to M'$ to be the unique term in $\mathbb{Y}$ such that $\mathbb{F}(\mathfrak{J}_x(f)) = \mathfrak{J}_x(\mathbb{F}(f))$ in $\mathbb{X}$. By definition of $\mathfrak{r}_{\tilde{A}}$, we have that

$$\mathbb{F}(\mathfrak{r}_x^\Gamma; \mathfrak{J}_x(f)) = \mathfrak{r}_x^{\mathbb{F}(\Gamma)}; \mathfrak{J}_x(\mathbb{F}(f)) = \mathbb{F}(f).$$

Hence $M = M'$ and $f = \mathfrak{r}_x^\Gamma; \mathfrak{J}_x(f)$. On the other hand, whenever $g : \Gamma \oplus_x \mathcal{H}_A \to M$ in $\mathbb{Y}$ we have that

$$\mathbb{F}(\mathfrak{J}_x(\mathfrak{r}_x^\Gamma; g)) = \mathfrak{J}_x(\mathbb{F}(\mathfrak{r}_x; g)) = \mathfrak{r}_x^{\mathbb{F}(\Gamma)}; \mathbb{F}(g) = \mathbb{F}(g)$$

and so $\mathfrak{J}_x(\mathfrak{r}_x; g) = g$. Hence, we have defined the data of a homomorphism type at $\tilde{A}$. □

Corollary 3.3.0.7. *Whenever* $\mathbb{X}$ *has strict homomorphism types and* $\mathbb{F} : \mathbb{X} \to \mathrm{SpanSet}$ *is an algebra of* $\mathbb{X}$*, the globular multicategory of elements* $\mathrm{el}(\mathbb{F})$ *has strict homomorphism types.*

Proof. This follows from the fact that the canonical projection $\pi_{\mathbb{F}} : \mathrm{el}(\mathbb{F}) \to \mathbb{X}$ is a discrete opfibration. □

Remark 3.3.0.8. Proposition 3.3.0.6 can alternatively be proved by observing that the globular multicategory of pointed sets $\mathrm{SpanSet}_\star$ has strict homomorphism types, the universal discrete opfibration $\mathrm{SpanSet}_\star \to \mathrm{SpanSet}$ preserves homomorphism types, and the forgetful functor $U_{\mathcal{H}} : \mathrm{GlobMult}_{\mathcal{H}} \to \mathrm{GlobMult}$ creates pullbacks. The result now follows, up-to-size-constraints from Remark 2.6.1.7. See also Proposition 4.2.1.7 and Corollary 4.2.1.8.

Example 3.3.0.9. Combining Corollary 3.3.0.7, and Example 3.3.0.4, whenever $\mathcal{C} : \mathbb{1} \to \mathrm{SpanSet}$ is a strict ω-category, the globular multicategory $\mathrm{el}(\mathcal{C})$ has strict homomorphism types.

Example 3.3.0.10. Suppose that $\mathcal{C} : \mathbb{1} \to \mathrm{SpanSet}$ is a strict ω-category. Then the vertical globular multicategory $\mathbb{V}(\mathcal{C})$ has strict homomorphism types. For each n-type $\mathcal{H}^n_A$ in $\mathbb{V}(\mathcal{C})$ we define $\mathcal{H}_{\mathcal{H}^n_A} = \mathcal{H}^{n+1}_A$. We define $\mathfrak{r}_{\mathcal{H}^n_A} : \mathcal{H}^n_A \to \mathcal{H}^{n+1}_A$ so that

$$\overline{\mathfrak{r}_{\mathcal{H}^n_A}} = \mathrm{id}^{n+2}_A,$$

and so that $\mathbf{o}_{\mathfrak{r}_{\mathcal{H}^n_A}}$ is the unique term $n+1 \to n+2$ in $\mathbb{1}$. The unit laws of $\mathcal{C}$ now imply that this data define a homomorphism type. In fact, this is the objects-part of a fully-faithful functor $\mathbb{V} : \mathrm{Str}\,\omega\text{-}\mathrm{Cat} \to \mathrm{GlobMult}_{\mathcal{H}}$.

Example 3.3.0.11. Analogous results hold for n-globular multicategories and n-categories, when n is finite.

Remark 3.3.0.12. Suppose that $k < n \leq \omega$. Restricting the truncation functor to the subcategory of globular multicategories with homomorphism types, we obtain a functor $\mathrm{tr}_k : n\text{ - }\mathrm{GlobMult}_{\mathcal{H}} \to k\text{ - }\mathrm{GlobMult}_{\mathcal{H}}$. The following diagram commutes:

$$\begin{tikzcd}
n\text{ - }\mathrm{GlobMult}_{\otimes} \arrow[r, "\mathrm{tr}_k"] \arrow[d, "U_{\otimes}"'] & k\text{ - }\mathrm{GlobMult}_{\otimes} \arrow[d, "U_{\otimes}"] \\
n\text{ - }\mathrm{GlobMult}_{\mathcal{H}} \arrow[r, "\mathrm{tr}_k"] \arrow[d, "U_{\mathcal{H}}"'] & k\text{ - }\mathrm{GlobMult}_{\mathcal{H}} \arrow[d, "U_{\mathcal{H}}"] \\
n\text{ - }\mathrm{GlobMult} \arrow[r, "\mathrm{tr}_k"'] & m\text{ - }\mathrm{GlobMult}
\end{tikzcd}$$

That is, truncation functors commute with the functors forgetting representability, and homomorphism types. Additionally, the truncation functor tr_k has a fully faithful left adjoint $L_{\mathrm{tr}_k} : k\text{ - }\mathrm{GlobMult}_{\mathcal{H}} \to n\text{ - }\mathrm{GlobMult}_{\mathcal{H}}$. We typically identify k-globular multicategories with strict homomorphism types with n-dimensional globular multicategories with strict homomorphism types using L_{tr_k}. However, there is a subtlety to this identification. Let $U_{\mathcal{H}} : \mathrm{GlobMult}_{\mathcal{H}} \longrightarrow \mathrm{GlobMult}$ be the functor forgetting homomorphism types. Then, the following diagram of left adjoints does *not* commute:

$$\begin{tikzcd}
k\text{ - }\mathrm{GlobMult}_{\mathcal{H}} \arrow[r, "L_{\mathrm{tr}_k}"] \arrow[d, "U_{\mathcal{H}}"'] \arrow[dr, phantom, "\not\circlearrowright"] & n\text{ - }\mathrm{GlobMult}_{\mathcal{H}} \arrow[d, "U_{\mathcal{H}}"] \\
k\text{ - }\mathrm{GlobMult} \arrow[r, "L_{\mathrm{tr}_k}"'] & n\text{ - }\mathrm{GlobMult}
\end{tikzcd}$$

In order to see this, suppose that $\mathbb{X}$ is a globular multicategory with homomorphism types. Then, for any $m > k$, the globular multicategory $L_{\mathrm{tr}_k} U_{\mathcal{H}} \mathbb{X}$ has no m-types, while an m-type in the globular multicategory $L_{\mathrm{tr}_k} \mathbb{X}$ is an iterated homomorphism type of a k-type in $\mathbb{X}$. On the other hand, we do have the following commutative diagram:

$$\begin{tikzcd}
k\text{ - GlobMult}_{\otimes} \arrow[r, "L_{\mathrm{tr}_k}"] \arrow[d, "U_{\otimes}"'] & n\text{ - GlobMult}_{\otimes} \arrow[d, "U_{\otimes}"] \\
k\text{ - GlobMult}_{\mathcal{H}} \arrow[r, "{L_{\mathrm{tr}_k}}'"'] & n\text{ - GlobMult}_{\mathcal{H}}
\end{tikzcd}$$

In contrast, the corresponding result does not hold for plain globular multicategories; the following square does not commute:

$$\begin{tikzcd}
k\text{ - GlobMult}_{\otimes} \arrow[r, "L_{\mathrm{tr}_k}"] \arrow[d, "U_{\otimes}"'] \arrow[dr, phantom, "\not\circlearrowleft"] & n\text{ - GlobMult}_{\otimes} \arrow[d, "U_{\otimes}"] \\
k\text{ - GlobMult} \arrow[r, "L_{\mathrm{tr}_k}"'] & n\text{ - GlobMult}
\end{tikzcd}$$

This is one way in which globular multicategories with homomorphism types are more similar to representable globular multicategories than ordinary globular multicategories. As further examples of this phenomenon, the globular multicategory of elements construction and the vertical construction commute with $L_{\mathrm{tr}_k} : k\text{-GlobMult}_{\mathcal{H}} \to n\text{ - GlobMult}_{\mathcal{H}}$. However, they do not commute with $L_{\mathrm{tr}_k} : k\text{ - GlobMult} \to n\text{ - GlobMult}$.

3.3.1 Free Results

Let $U_{\mathcal{H}} : \mathrm{GlobMult}_{\mathcal{H}} \longrightarrow \mathrm{GlobMult}$ be the functor forgetting homomorphism types. We will see in this section that $U_{\mathcal{H}}$ satisfies many good properties by general results.

Proposition 3.3.1.1. *The* 2*-categories* GlobMult *and* $\mathrm{GlobMult}_{\mathcal{H}}$ *are locally finitely presentable. Furthermore, the strict* 2*-functor* $U_{\mathcal{H}}$ *has a strict left* 2*-adjoint.*

$$\begin{tikzcd}
\mathrm{GlobMult} \arrow[r, bend left, "L_{\mathcal{H}}"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult}_{\mathcal{H}} \arrow[l, bend left, "U_{\mathcal{H}}"]
\end{tikzcd}$$

Proof. Enriched Gabriel-Ulmer duality tells us that there is an equivalence of 2-categories between the 2-category of finitely complete categories (or equivalently essentially algebraic theories) with limit preserving functors between them and the 2-category of locally finitely presentable categories with finitary right adjoint functors between them. This result is proved in the enriched setting in [29].

The definition of GlobMult (and $\text{GlobMult}_{\mathcal{H}}$) exhibits the category of globular multicategories (with homomorphism types) as the category of models of an essentially algebraic theory. We can view these essentially algebraic theories as being Cat-enriched by considering $\text{Type}(\mathbb{X})(n)$ to be a category and not just a set. With this convention, the 2-category GlobMult (or $\text{GlobMult}_{\mathcal{H}}$) is the category of models of a Cat-enriched essentially algebraic theory. Furthermore, $U_{\mathcal{H}}$ is a functor forgetting some of this essentially algebraic structure, namely the homomorphism types. Hence, applying Cat-enriched Gabriel-Ulmer duality, we immediately obtain the desired result. □

Another useful property of $U_{\mathcal{H}}$ can also be obtained from this analysis:

Proposition 3.3.1.2. *The forgetful functor $U_{\mathcal{H}}$ is conservative (reflects isomorphisms).*

Proof. This is true in general of functors forgetting essentially algebraic data. We will describe this result explicitly for clarity. Suppose that $f : \mathbb{X} \to \mathbb{Y}$ is a homomorphism of globular multicategories with homomorphism types and that $g : \mathbb{Y} \to \mathbb{X}$ is an inverse homomorphism (not necessarily preserving homomorphism types). Then

$$\begin{aligned} g(\mathcal{H}_A) = g(\mathcal{H}_{fgA}) &= g(f(\mathcal{H}_{g_A})) &&= \mathcal{H}_{gA} \\ g(\mathfrak{r}_A) = g(\mathfrak{r}_{fgA}) &= g(f(\mathfrak{r}_{g_A})) &&= \mathfrak{r}_{gA} \end{aligned}$$

and so g also preserves homomorphism types. □

3.4 The Strict Higher Modules Construction

While the left adjoint of $U_{\mathcal{H}}$: GlobMult → GlobMult exists by a general argument, the construction of the right adjoint is more involved. Our aim in this section is to describe how a higher-dimensional analogue of the monoids and modules construction allows us systematically to construct "higher category-like" objects together with higher notions of transformation and module. We will prove the following theorem:

Theorem 3.4.0.1. *The forgetful functor $U_{\mathcal{H}}$: $\text{GlobMult}_{\mathcal{H}} \to \text{GlobMult}$ has a right adjoint.*

Crutwell and Shulman [17] have shown the 1-dimensional version of this result:

Definition 3.4.0.2. A *monoid* in a 1-globular multicategory $\mathbb{X}$, consists of a 0-type A together with a 1-type $\mathcal{H}_A : A \nrightarrow A$, a multiplication 1-term

$$\begin{tikzcd}
A \arrow[r, "\mathcal{H}_A", "\shortmid" marking] \arrow[d, equal] & A \arrow[r, "\mathcal{H}_A", "\shortmid" marking] \arrow[d, phantom, "\Downarrow \mathfrak{m}_A"] & A \arrow[d, equal] \\
A \arrow[rr, "\mathcal{H}_A"', "\shortmid" marking] & & A
\end{tikzcd}$$

and a unit 1-term

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathfrak{r}_A"] & A \arrow[d, equal] \\
A \arrow[r, "\mathcal{H}_A"', "\shortmid" marking] & A
\end{tikzcd}$$

satisfying associativity and unit laws. A *module* is a 1-type $M : A \nrightarrow B$ with actions on the left and right by the monoids A and B. These are 1-terms of the form:

$$\begin{tikzcd}
A \arrow[r, "\mathcal{H}_A", "\shortmid" marking] \arrow[d, equal] & A \arrow[r, "M", "\shortmid" marking] \arrow[d, phantom, "\Downarrow \lambda_M"] & B \arrow[d, equal] & A \arrow[r, "M", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "H_A", "\shortmid" marking] \arrow[d, phantom, "\Downarrow \rho_M"] & B \arrow[d, equal] \\
A \arrow[rr, "M"', "\shortmid" marking] & & B & A \arrow[rr, "M"', "\shortmid" marking] & & B
\end{tikzcd}$$

These 1-terms must be compatible with the multiplication and units of A and B. Monoids and modules are the types of a 1-globular multicategory $\mathrm{Mod}\,\mathbb{X}$, and this is the objects-part of a strict 2-functor $\mathrm{Mod} : 1\text{ - GlobMult} \to 1\text{ - GlobMult}_{\mathcal{H}}$.

Theorem 3.4.0.3 ([17]). *The monoids and modules functor* $\mathrm{Mod} : 1\text{ - GlobMult} \to 1\text{ - GlobMult}_{\mathcal{H}}$ *is right adjoint to the functor* $U_{\mathcal{H}} : 1\text{ - GlobMult} \to 1\text{ - GlobMult}$ *that forgets homomorphism type data.*

Hence, in order to generalize this result to higher dimensions, we describe higher dimensional versions of monoids and modules, and exhibit a higher dimensional modules construction $\mathrm{Mod} : \ \mathrm{GlobMult} \to \ \mathrm{GlobMult}_{\mathcal{H}}$ as the right adjoint of $U_{\mathcal{H}}$. We first provide an informal overview of the notions of higher modules and their homomorphisms. Then we give a more detailed account of this construction and its universal property.

3.4.1 Overview

For each n, we refer to an n-type in $\mathrm{Mod}\,\mathbb{X}$ as an *n-module in* $\mathbb{X}$. A 1-module, $M : A \nrightarrow B$ in $\mathbb{X}$ can be acted on by its 0-source and 0-target A and B. These actions amount to terms whose source contexts are of the following form:

$$A \xrightarrow{\mathcal{H}_A} A \xrightarrow{M} B, \qquad A \xrightarrow{M} B \xrightarrow{\mathcal{H}_B} B.$$

More generally, n-modules can be acted on by their k-dimensional source and target modules for all $k < n$. For example, a 2-module O, depicted as

$$\begin{tikzcd}
A \arrow[r, bend left=40, "M", "\shortmid"{marking}, ""{name=U, below}] \arrow[r, bend right=40, "N"', "\shortmid"{marking}, ""{name=D}] & B
\arrow[Rightarrow, from=U, to=D, "O", "\shortmid"{marking}]
\end{tikzcd}$$

can be acted on by A, B, M and N. Associated to these two actions are multiplication terms whose source contexts are

$$\begin{tikzcd}
A \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] & A \arrow[r, bend left=40, "M", "\shortmid"{marking}, ""{name=U, below}] \arrow[r, bend right=40, "N"', "\shortmid"{marking}, ""{name=D}] & B
\arrow[Rightarrow, from=U, to=D, "O", "\shortmid"{marking}]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, bend left=40, "M", "\shortmid"{marking}, ""{name=U, below}] \arrow[r, bend right=40, "N"', "\shortmid"{marking}, ""{name=D}] & B \arrow[r, "\mathcal{H}_B", "\shortmid"{marking}] & B
\arrow[Rightarrow, from=U, to=D, "O", "\shortmid"{marking}]
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, bend left=60, "M", "\shortmid"{marking}, ""{name=U, below}] \arrow[r, "M"{description}, ""{name=C}] \arrow[r, bend right=60, "N"', "\shortmid"{marking}, ""{name=D}] & B
\arrow[Rightarrow, from=U, to=C, "\mathcal{H}_M", "\shortmid"{marking}]
\arrow[Rightarrow, from=C, to=D, "O", "\shortmid"{marking}]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, bend left=60, "M", "\shortmid"{marking}, ""{name=U, below}] \arrow[r, "N"{description}, ""{name=C}] \arrow[r, bend right=60, "N"', "\shortmid"{marking}, ""{name=D}] & B
\arrow[Rightarrow, from=U, to=C, "O", "\shortmid"{marking}]
\arrow[Rightarrow, from=C, to=D, "\mathcal{H}_N", "\shortmid"{marking}]
\end{tikzcd}$$

respectively. We refer to n-terms in $\operatorname{Mod}\mathbb{X}$ as *n-module homomorphisms*. Like the module homomorphisms of the 1-dimensional monoids and modules construction, higher module homomorphisms satisfy *equivariance laws*. For example, given a homomorphism f with the source context

$$\begin{tikzcd}
A \arrow[r, bend left=60, "M", "\shortmid"{marking}, ""{name=U, below}] \arrow[r, "N"{description}, ""{name=C}] \arrow[r, bend right=60, "O"', "\shortmid"{marking}, ""{name=D}] & B \arrow[r, bend left=40, "P", "\shortmid"{marking}, ""{name=U2, below}] \arrow[r, bend right=40, "Q"', "\shortmid"{marking}, ""{name=D2}] & C,
\arrow[Rightarrow, from=U, to=C, "R", "\shortmid"{marking}]
\arrow[Rightarrow, from=C, to=D, "S", "\shortmid"{marking}]
\arrow[Rightarrow, from=U2, to=D2, "T", "\shortmid"{marking}]
\end{tikzcd}$$

there are two ways of building terms out of f and actions involving $\mathcal{H}_B$: one is induced by the actions of $\mathcal{H}_B$ on R and S, while the other is induced by the action of $\mathcal{H}_B$ on T. We require that these two terms agree. Homomorphisms can be composed because given composable homomorphisms f and g, the equivariance laws of f and g can be used to construct the equivariance laws of the composite $f; g$.

3.4.2 Level-wise Modules Constructions

Our tactic for making this description precise will be to first describe the actions of l-types on n-modules separately for each l, and then later to combine these *level-wise modules constructions.*

Definition 3.4.2.1. Let $l \geq 0$. A *globular multicategory with homomorphism types at level* l is a globular multicategory $\mathbb{X}$ together with a choice of homomorphism $(l+1)$-type for each l-type in $\mathbb{X}$. Let $S \subseteq [\mathbf{d}]$. We denote by $\mathrm{GlobMult}^{S}_{\mathcal{H}}$ the category of globular multicategories with homomorphism types at level l for each $l \in S$.

Fix $l \geq 0$. Let $U^{\{l\}}_{\mathcal{H}} : \mathrm{GlobMult}^{\{l\}}_{\mathcal{H}} \to \mathrm{GlobMult}$ be the functor forgetting homomorphism types at level l. We will define a functor $\mathrm{Mod}^{\{l\}} : \mathrm{GlobMult} \to \mathrm{GlobMult}^{\{l\}}_{\mathcal{H}}$ such that we have an adjunction of the following form:

$$\begin{tikzcd}
\mathrm{GlobMult}^{\{l\}}_{\mathcal{H}} \arrow[r, bend left, "U^{\{l\}}_{\mathcal{H}}"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult} \arrow[l, bend left, "\mathrm{Mod}^{\{l\}}"] .
\end{tikzcd}$$

This functor behaves much like the 1-dimensional monoids and modules construction. Indeed, the monoids and modules construction on virtual double categories is easily seen to be a special case of the construction presented here.

Let $\mathbb{X}$ be a globular multicategory. We now define a globular multicategory $\mathrm{Mod}^{\{l\}}\,\mathbb{X} \in \mathrm{GlobMult}^{\{l\}}$. Roughly speaking, the types of $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ are defined so that:

- When $n < l$, an n-type in $\mathrm{Mod}^{\{l\}}$ is an n-type in $\mathbb{X}$.
- When $n = l$, an n-type in $\mathrm{Mod}^{\{l\}}$ is an l-type A in $\mathbb{X}$ together with an $(l+1)$-type $\mathcal{H}_A$, and associative and unital *multiplication and unit terms.* Thus, l-types are *monoids.*
- When $n > l$, an n-type in $\mathrm{Mod}^{\{l\}}$ is an n-type M in $\mathbb{X}$ together with *actions* of $s_l M$ and $t_l M$ on M. These actions satisfy axioms saying that M is a *bimodule* over its l-source and l-target.

The terms are defined so that:

- When $n < l$, an n-term is just an n-term in $\mathbb{X}$.
- When $n = l$, an n-term $f : \Gamma \to A$ is an n-term of $\mathbb{X}$ respecting the multiplication of the l-types in Γ and A. That is, an n-term is a *monoid homomorphism*
- When $n > l$, an n-term is an n-term of $\mathbb{X}$ satisfying certain *equivariance laws.* That is, an n-term is a *module homomorphism.*

We will make each of these cases precise inductively. First, we consider the most straightforward case.

Definition 3.4.2.2. Suppose that $n < l$. Then, an n-type $M : A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ consists of an n-type $M_0 : M_0 \relbar\joinrel\mapstochar\joinrel\rightarrow B_0$ in $\mathbb{X}$. An n-term $f : \Gamma \to M, sf \relbar\joinrel\mapstochar\joinrel\rightarrow tf$ in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ is an n-term $f_0 : \Gamma_0 \to M_0\,(sf)_0 \relbar\joinrel\mapstochar\joinrel\rightarrow (tf)_0$ in $\mathbb{X}$. Composition of n-terms is composition in $\mathbb{X}$.

Next we consider the case where $n = l$.

Definition 3.4.2.3. An *l-monoid in* $\mathbb{X}$ consists of an l-type $M_0 : A_0 \relbar\joinrel\mapstochar\joinrel\rightarrow B_0$ in $\mathbb{X}$ together with:

- An $(l+1)$-type $\mathcal{H}_M : M_0 \to M_0$ in $\mathbb{X}$
- A multiplication $(l+1)$-term $\mathfrak{m}_M : \mathcal{H}_M \odot_l \mathcal{H}_M \to \mathcal{H}_{M_0}, \mathrm{id}_{M_0} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathrm{id}_M$ in $\mathbb{X}$
- A unit $(l+1)$-term $\mathfrak{r}_M : M \to \mathcal{H}_M, \mathrm{id}_M \relbar\joinrel\mapstochar\joinrel\rightarrow \mathrm{id}_M$ in $\mathbb{X}$

We require that multiplication is associative and unital; that is

$$(\mathfrak{m}_M \odot_l \mathrm{id}_{\mathcal{H}_M}); \mathfrak{m}_M = (\mathrm{id}_{\mathcal{H}_M} \odot_l \mathfrak{m}_M); \mathfrak{m}_M,$$

and

$$(\mathfrak{r}_M \odot_l \mathrm{id}_{\mathcal{H}_M}); \mathfrak{m}_M = \mathrm{id}_{\mathcal{H}_M} = (\mathrm{id}_{\mathcal{H}_M} \odot_l \mathfrak{r}_M); \mathfrak{m}_M.$$

Example 3.4.2.4. A 0-monoid in $\mathbb{X}$ consists of a 0-type A_0, a 1-type $\mathcal{H}_A : A_0 \relbar\joinrel\mapstochar\joinrel\rightarrow A_0$, a multiplication 1-term

$$\begin{tikzcd}
A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \mathfrak{m}_A"] & A_0 \arrow[d, equal] \\
A_0 \arrow[rr, "\mathcal{H}_A"', "\shortmid"{marking}] & & A_0
\end{tikzcd}$$

and a unit 1-term

$$\begin{tikzcd}
A_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathfrak{r}_A"] & A_0 \arrow[d, equal] \\
A_0 \arrow[r, "\mathcal{H}_A"', "\shortmid"{marking}] & A_0
\end{tikzcd}$$

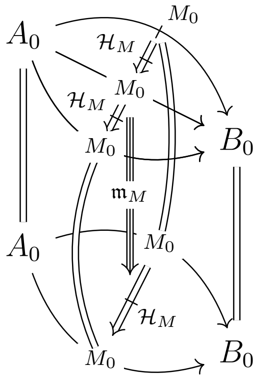
such that

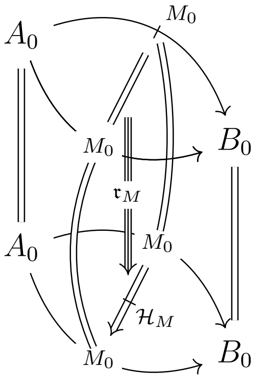
$$\begin{tikzcd}
A_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathfrak{r}_A"] & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & A_0 \arrow[d, equal] \\
A_0 \arrow[r, "\mathcal{H}_A"'] \arrow[d, equal] & A_0 \arrow[r, "\mathcal{H}_A"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \mathfrak{m}_A"] & A_0 \arrow[d, equal] \\
A_0 \arrow[rr, "\mathcal{H}_A"', "\shortmid"{marking}] & & A_0
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & A_0 \arrow[d, equal] \\
A_0 \arrow[r, "\mathcal{H}_A"', "\shortmid"{marking}] & A_0
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, "\mathcal{H}_A"] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathfrak{r}_A"] & A_0 \arrow[d, equal] \\
A_0 \arrow[r, equal, "\mathcal{H}_A"'] \arrow[d, equal] & A_0 \arrow[r, "\mathcal{H}_A"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \mathfrak{m}_A"] & A_0 \arrow[d, equal] \\
A_0 \arrow[rr, "\mathcal{H}_A"', "\shortmid"{marking}] & & A_0
\end{tikzcd}$$

and

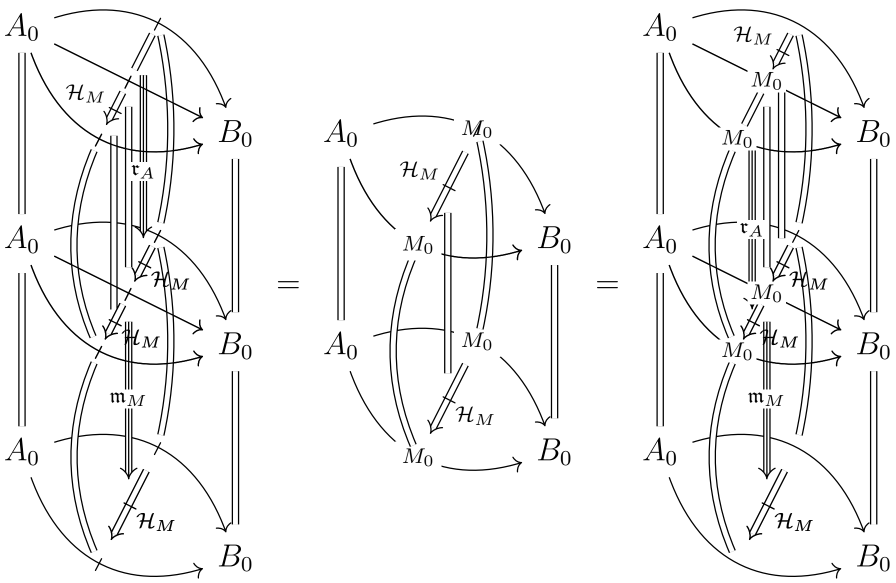
$$\begin{tikzcd}
A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equals] & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \mathfrak{m}_A"] & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\Vert"] & A_0 \arrow[d, equals] & & A_0 \arrow[r, "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\Vert"] & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equals] & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \mathfrak{m}_A"] & A_0 \arrow[d, equals] \\
A_0 \arrow[rr, "\mathcal{H}_A"', "\shortmid"{marking}] \arrow[d, equals] & & A_0 \arrow[r, "\mathcal{H}_A"', "\shortmid"{marking}] \arrow[dl, phantom, "\Downarrow \mathfrak{m}_A"] & A_0 \arrow[d, equals] & = & A_0 \arrow[r, "\mathcal{H}_A"', "\shortmid"{marking}] \arrow[d, equals] & A_0 \arrow[rr, "\mathcal{H}_A"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \mathfrak{m}_A"] & & A_0 \arrow[d, equals] \\
A_0 \arrow[rrr, "\mathcal{H}_A"', "\shortmid"{marking}] & & & A_0 & & A_0 \arrow[rrr, "\mathcal{H}_A"', "\shortmid"{marking}] & & & A_0
\end{tikzcd}$$

Example 3.4.2.5. A 1-monoid in $\mathbb{X}$ consists of a 1-type $M_0 : A_0 \to B_0$ together with a 2-type $\mathcal{H}_M : M_0 \nrightarrow M_9$ and multiplication and unit 2-terms

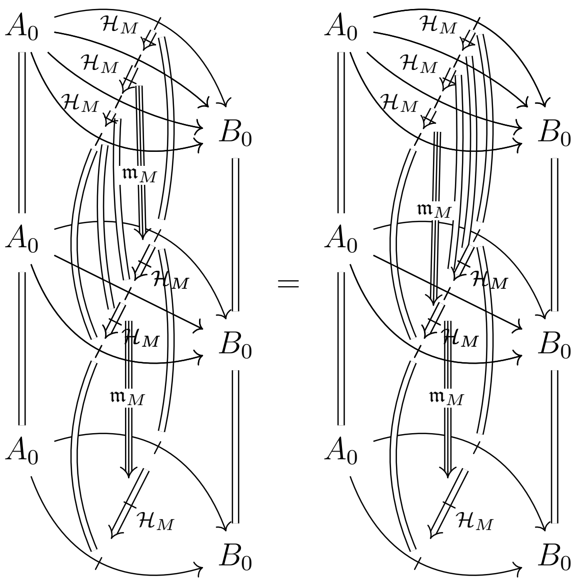




such that



and



Definition 3.4.2.6. An l-type in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ is an l-monoid in $\mathbb{X}$.

Suppose that Γ is an l-context in $\mathrm{Mod}^{\{l\}}\,X$. Then, by construction, there is an underlying context $\Gamma_0 = \bigodot_{i\in\Gamma}(\Gamma_i)_0$ in $\mathbb{X}$. Suppose that $S = \{x_1 : A_1, \ldots, x_k : A_k\}$ is a set of l-variables in Γ. Then, we define the $(l+1)$-context $\mathcal{H}^\Gamma_S : \Gamma \nrightarrow \Gamma$ by

$$\mathcal{H}^\Gamma_S = \Gamma_0 \oplus_S \mathcal{H} = \Gamma_0 \oplus_{x_1} \mathcal{H}_{A_1} \oplus_{x_2} \cdots \oplus_{x_k} \mathcal{H}_{A_k}.$$

Similarly, we define the $(l+1)$-terms $\mathfrak{r}^\Gamma_S : \Gamma \to \mathcal{H}^\Gamma_S,\ \mathrm{id}_\Gamma \nrightarrow \mathrm{id}_\Gamma$, and $\mathfrak{m}^\Gamma_S : \mathcal{H}^\Gamma_S \odot_l \mathcal{H}^\Gamma_S \to \mathcal{H}^\Gamma_S,\ \mathrm{id}_\Gamma \nrightarrow \mathrm{id}_\Gamma$ by

$$\mathfrak{r}^\Gamma_S = \mathrm{id}_\Gamma \oplus_{x_1} \mathfrak{r}_{A_1} \oplus_{x_2} \cdots \oplus_{x_k} \mathfrak{r}_{A_k}, \qquad m^\Gamma_S = \mathrm{id}_\Gamma \oplus_{x_1} \mathfrak{m}_{A_1} \oplus_{x_2} \cdots \oplus_{x_k} \mathfrak{m}_{A_k}, .$$

In the the maximal case, when $S = \Gamma(l)$, we define

$$\mathcal{H}_\Gamma = \mathcal{H}^\Gamma_{\Gamma(l)}, \qquad \mathfrak{r}_\Gamma = \mathfrak{r}^\Gamma_{\Gamma(l)}, \qquad \mathfrak{m}_\Gamma = \mathfrak{m}^\Gamma_{\Gamma(l)}.$$

We denote the complement of a set $S \subseteq \Gamma(l)$ by $\tilde{S}$. When $S = \widetilde{\{x\}}$, we write $\mathfrak{r}^\Gamma_{\tilde{x}} = \mathfrak{r}^\Gamma_{\widetilde{\{x\}}}$.

Definition 3.4.2.7. An *l-monoid homomorphism $f : \Gamma \to M$, $sf \nrightarrow tf$ in* $\mathbb{X}$ consists of an l-term $f_0 : \Gamma_0 \to M_0$, $(sf)_0 \nrightarrow (tf)_0$ in $\mathbb{X}$, together with an $(l+1)$-term

$$\mathcal{H}_f : \mathcal{H}_\Gamma \longrightarrow \mathcal{H}_M, \quad f_0 \nrightarrow f_0,$$

in $\mathbb{X}$. This term must respect the multiplication and unit terms of Γ and M:

$$\mathfrak{m}_\Gamma ; \mathcal{H}_f = (\mathcal{H}_f \odot_l \mathcal{H}_f) ; \mathfrak{m}_M, \qquad \mathfrak{r}_\Gamma ; \mathcal{H}_f \ = f_0 ; \mathfrak{r}_M.$$

Example 3.4.2.8. A 0-monoid homomorphism in $\mathbb{X}$ consists of a 0-term $f_0 : A \to B$ together with a 1-term

$$\begin{tikzcd}
A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow \mathcal{H}_f"] & A_0 \arrow[d, "f_0"] \\
B_0 \arrow[r, "\mathcal{H}_B"', "\shortmid"{marking}] & B_0
\end{tikzcd}$$

such that

$$\begin{tikzcd}
A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, Rightarrow, "\mathfrak{m}_A"] & A_0 \arrow[d, equal] \\
A_0 \arrow[rr, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, "f_0"'] & {} \arrow[d, Rightarrow, "\mathcal{H}_f"] & A_0 \arrow[d, "f_0"] \\
B_0 \arrow[rr, "\mathcal{H}_B"', "\shortmid"{marking}] & {} & B_0
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\Downarrow \mathcal{H}_f"] & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, dashed, "f_0"] \arrow[dr, phantom, "\Downarrow \mathcal{H}_f"] & A_0 \arrow[d, "f_0"] \\
B_0 \arrow[r, "\mathcal{H}_B"', "\shortmid"{marking}] \arrow[d, equal] & B_0 \arrow[r, "\mathcal{H}_B"', "\shortmid"{marking}] \arrow[d, Rightarrow, "\mathfrak{m}_B"] & B_0 \arrow[d, equal] \\
B_0 \arrow[rr, "\mathcal{H}_B"', "\shortmid"{marking}] & {} & B_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
{} & A_0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, Rightarrow, "\mathfrak{r}_A"] & {} \\
A_0 \arrow[rr, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, "f_0"'] & {} \arrow[d, Rightarrow, "\mathcal{H}_f"] & A_0 \arrow[d, "f_0"] \\
B_0 \arrow[rr, "\mathcal{H}_B"', "\shortmid"{marking}] & {} & B_0
\end{tikzcd}
=
\begin{tikzcd}
{} & A_0 \arrow[d, "f_0"] & {} \\
{} & B_0 \arrow[dl, equal] \arrow[dr, equal] \arrow[d, Rightarrow, "\mathfrak{r}_B"] & {} \\
B_0 \arrow[rr, "\mathcal{H}_B"', "\shortmid"{marking}] & {} & B_0
\end{tikzcd}$$

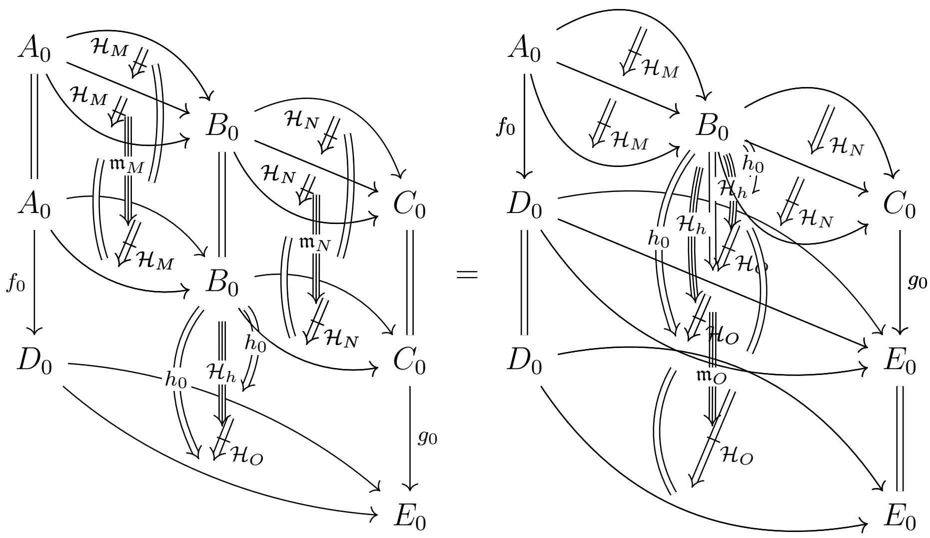
Example 3.4.2.9. Suppose that $M : A \nrightarrow B$, $N : B \nrightarrow C$ and $O : D \to E$ are 1-monoids in $\mathbb{X}$. Then a 1-monoid homomorphism $h : M \odot_0 N \to O, f \nrightarrow g$ consists of a 1-term

$$\begin{tikzcd}
A_0 \arrow[r, "M_0", "\shortmid"{marking}] \arrow[d, "f_0"'] & B_0 \arrow[r, "N_0", "\shortmid"{marking}] \arrow[d, Rightarrow, "h_0"] & C_0 \arrow[d, "g_0"] \\
D_0 \arrow[rr, "O_0"', "\shortmid"{marking}] & {} & E_0
\end{tikzcd}$$

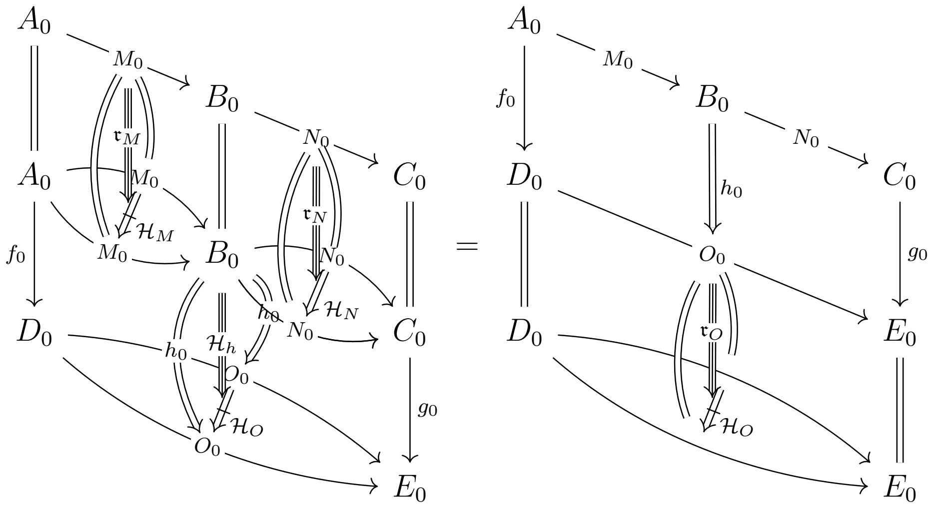
together with a 2-term

$$\begin{tikzcd}
A_0 \arrow[dr, bend left, "M_0"] \arrow[dr, bend right, "M_0"'] \arrow[dr, phantom, "\Downarrow \mathcal{H}_M"] \arrow[dd, "f_0"'] & {} & {} \\
{} & B_0 \arrow[dr, bend left, "N_0"] \arrow[dr, bend right, "N_0"'] \arrow[dr, phantom, "\Downarrow \mathcal{H}_N"] \arrow[dd, phantom, "\overset{h_0}{\Downarrow} \ \Rrightarrow \mathcal{H}_h \ \overset{h_0}{\Downarrow}"] & {} \\
D_0 \arrow[ddrr, bend left, "O_0"] \arrow[ddrr, bend right, "O_0"'] \arrow[ddrr, phantom, "\Downarrow \mathcal{H}_O"] & {} & C_0 \arrow[dd, "g_0"] \\
{} & {} & {} \\
{} & {} & E_0
\end{tikzcd}$$

such that



and



Definition 3.4.2.10. An l-term in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ is an l-monoid homomorphism.

Suppose that $f : \Gamma \to \Delta$ is an l-substitution in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. Then, we define $f_0 : \Gamma_0 \to \Delta_0$ by $f_0 = \bigodot_{i \in \Delta} (f_i)_0$. We define $\mathcal{H}_f : \mathcal{H}_\Gamma \to \mathcal{H}_\Delta,\ f_0 \nrightarrow f_0$ so that, for each $x \in \mathcal{H}_\Delta$,

$$(\mathcal{H}_f)_x = \begin{cases} \mathcal{H}_{f_x} & \text{if } \dim x = l + 1 \\ (f_x)_0 & \text{if } \dim x \neq l \end{cases}$$

Composition of l-terms is defined by:

$$(f; g)_0 = f_0; g_0, \qquad \mathcal{H}_{f;g} \ = \mathcal{H}_f; \mathcal{H}_g.$$

Given an l-monoid A in $\mathbb{X}$, we define the identity l-monoid homomorphism by

$$(\mathrm{id}_A)_0 = \mathrm{id}_{A_0}, \qquad \mathcal{H}_{\mathrm{id}_A} = \mathrm{id}_{\mathcal{H}_A}\,.$$

These data make $\mathrm{Mod}^{\{l\}}\,\mathbb{X}(l)$ a category, and they are easily seen to respect the globular structure of $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$.

Suppose $f : \Gamma \to M$ is an l-monoid homomorphism, and suppose that $S \subseteq \Gamma_0(l)$. Then, we define the term $\mathcal{H}_{f,S} : \mathcal{H}_{f,S} : \mathcal{H}^{\Gamma}_{S} \to \mathcal{H}_M,\ f_0 \nvrightarrow f_0$ by

$$\mathcal{H}_{f,S} = \mathfrak{r}^{\Gamma}_{\tilde{S}}; \mathcal{H}_f.$$

When S is a singleton $\{x\}$, we simply write $\mathcal{H}_{f,x}$.

Lemma 3.4.2.11. *Suppose that $f : \Gamma \to M$ is an l-term in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$, and that $S, T \subseteq \Gamma(l)$ are disjoint sets of l-variables. Then*

$$(\mathcal{H}_{f,S} \odot_l \mathcal{H}_{f,T}); \mathfrak{m}_M \ = \mathcal{H}_{f,S\cup T}$$

Proof. By the unit laws relating $\mathfrak{r}$ and $\mathfrak{m}$, for each variable $z \in \Gamma$, we have that

$$((\mathfrak{r}_{\tilde{S}} \odot_l \mathfrak{r}_{\tilde{T}}); \mathfrak{m}_\Gamma)_z = \begin{cases} (\mathrm{id}_z \odot_l \mathfrak{r}_z); \mathfrak{m}_z & \text{if } z \in S \\ (\mathfrak{r}_z \odot_l \mathrm{id}_z); \mathfrak{m}_z & \text{if } z \in T \\ (\mathfrak{r}_z \odot_l \mathfrak{r}_z); \mathfrak{m}_z & \text{if } \dim z = l \text{ and } z \notin S \cup T \\ \mathrm{id}_z & \text{otherwise} \end{cases}$$

$$= \begin{cases} \mathrm{id}_z & \text{if } z \in S \\ \mathrm{id}_z & \text{if } z \in T \\ \mathfrak{r}_z & \text{if } \dim z = l \text{ and } z \notin S \cup T \\ \mathrm{id}_z & \text{otherwise} \end{cases}$$

$$= \mathfrak{r}_{\widetilde{S\cup T}}$$

Hence, since $\mathcal{H}_f$ preserves multiplication, we have that

$$\begin{aligned} (\mathcal{H}_{f,S} \odot_l \mathcal{H}_{f,T}); \mathfrak{m}_M \ &= (\mathfrak{r}_{\tilde{S}} \odot_l \mathfrak{r}_{\tilde{T}}); (\mathcal{H}_f \odot_l \mathcal{H}_f); \mathfrak{m}_M \\ &= (\mathfrak{r}_{\tilde{S}} \odot_l \mathfrak{r}_{\tilde{T}}); \mathfrak{m}_\Gamma; \mathcal{H}_f \\ &= \mathfrak{r}_{\widetilde{S\cup T}}; \mathcal{H}_f \\ &= \mathcal{H}_{f,S\cup T} \end{aligned}$$

□

We now consider the n-types and -terms of $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ when $n > l$. This is by far the most complicated case.

Definition 3.4.2.12. An (n, l)*-module* M *in* $\mathbb{X}$ consists of an n-type M_0 in $\mathbb{X}$ together with the following data:

- When $n = l + 1$, we require a choice of l-monoids $s_l M, t_l M$ in $\mathbb{X}$ such that $(s_l M)_0 = s_l M_0$ and $(t_l M)_0 = t_l M_0$.
- When $n > l + 1$, we require a choice of $(n-1, l)$-modules sM, tM such that $(sM)_0 = sM_0$ and $(tM)_0 = tM_0$.
- We require actions

  $$\lambda^l_M : \mathcal{H}_{s_l M} \odot_l M_0 \longrightarrow M_0, \qquad \rho^l_M : M_0 \odot_l \mathcal{H}_{t_l M} \longrightarrow M_0$$

  of $s_l M$ and $t_l M$ on M such that when $n = l + 1$, we have that

  $$\lambda^l_M, \rho^l_M : \mathrm{id}_{sM} \twoheadrightarrow \mathrm{id}_{tM},$$

  and when $n > l + 1$, we have that

  $$\lambda^l_M : \lambda^l_{sM} \twoheadrightarrow \lambda^l_{tM}, \qquad \rho^l_M : \rho^l_{sM} \twoheadrightarrow \rho^l_{tM}.$$

  These actions must respect the multiplication and unit of M:

  $$(\mathfrak{r}_M \odot_l \mathrm{id}_M); \lambda^l_M = \mathrm{id}_M, \qquad (\mathrm{id}_M \odot_l \mathfrak{r}_M); \rho^l_M = \mathrm{id}_M,$$

  and

  $$(\mathfrak{m}_M \odot_l \mathrm{id}_M); \lambda^l_M = (\mathrm{id}_{\mathcal{H}_{s_l M}} \odot_l \lambda^l_M); \lambda^l_M,$$
  $$(\mathrm{id}_M \odot_l \mathfrak{m}_M); \rho^l_M = (\rho^l_M \odot_l \mathrm{id}_{\mathcal{H}_{t_l M}}); \rho^l_M.$$

  Furthermore, these actions must be compatible with each other:

  $$(\mathrm{id}_{\mathcal{H}_{s_l M}} \odot_l \rho^l_M); \lambda^l_M = (\lambda^l_M \odot_l \mathrm{id}_{\mathcal{H}_{t_l M}}); \rho^l_M.$$

Example 3.4.2.13. A $(1, 0)$-module consists of a 1-type

$$\begin{tikzcd} A_0 \arrow[r, "M_0", "|" marking] & B_0 \end{tikzcd}$$

such that A_0 and B_0 underlie 0-monoids, together with 1-terms

$$\begin{tikzcd} A_0 \arrow[r, "\mathcal{H}_A", "|" marking] \arrow[d, equal] & A_0 \arrow[r, "M_0", "|" marking] \arrow[d, Rightarrow, "\lambda^0_M"] & B_0 \arrow[d, equal] \\ A_0 \arrow[rr, "M_0"', "|" marking] & {} & B_0 \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[r, "M_0", "|" marking] \arrow[d, equal] & B_0 \arrow[r, "\mathcal{H}_B", "|" marking] \arrow[d, Rightarrow, "\rho^0_M"] & B_0 \arrow[d, equal] \\ A_0 \arrow[rr, "M_0"', "|" marking] & {} & B_0 \end{tikzcd}$$

The laws for the action λ^0_M say that

$$\begin{tikzcd}
A_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathfrak{r}_A"] & A_0 \arrow[r, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & B_0 \arrow[d, equal] \\
A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \lambda^l_M"] & A_0 \arrow[r, "M_0", "\shortmid"{marking}] & B_0 \arrow[d, equal] \\
A_0 \arrow[rr, "M_0"', "\shortmid"{marking}] & & B_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \mathfrak{m}_A"] & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] & A_0 \arrow[r, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & B_0 \arrow[d, equal] & & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \lambda^0_M"] & A_0 \arrow[r, "M_0", "\shortmid"{marking}] & B_0 \arrow[d, equal] \\
A_0 \arrow[rr, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow \lambda^0_M"] & & A_0 \arrow[r, "M_0", "\shortmid"{marking}] & B_0 \arrow[d, equal] & = & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow \lambda^0_M"] & A_0 \arrow[rr, "M_0", "\shortmid"{marking}] & & B_0 \arrow[d, equal] \\
A_0 \arrow[rrr, "M_0"', "\shortmid"{marking}] & & & B_0 & & A_0 \arrow[rrr, "M_0"', "\shortmid"{marking}] & & & B_0
\end{tikzcd}$$

The laws for the action ρ^0_M say that

$$\begin{tikzcd}
A_0 \arrow[r, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & B_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \mathfrak{r}_B"] & B_0 \arrow[d, equal] \\
A_0 \arrow[r, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho^l_M"] & B_0 \arrow[r, "\mathcal{H}_B", "\shortmid"{marking}] & B_0 \arrow[d, equal] \\
A_0 \arrow[rr, "M_0"', "\shortmid"{marking}] & & B_0
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & B_0 \arrow[d, equal] \\
A_0 \arrow[r, "M_0", "\shortmid"{marking}] & B_0
\end{tikzcd}
=
\begin{tikzcd}
A_0 \arrow[r, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & B_0 \arrow[d, equal] \\
A_0 \arrow[r, "M_0", "\shortmid"{marking}] & B_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
A_0 \arrow[r, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & B_0 \arrow[r, "\mathcal{H}_B", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \mathfrak{m}_B"] & B_0 \arrow[r, "\mathcal{H}_B", "\shortmid"{marking}] & B_0 \arrow[d, equal] & & A_0 \arrow[r, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho^0_M"] & B_0 \arrow[r, "\mathcal{H}_B", "\shortmid"{marking}] & B_0 \arrow[r, "\mathcal{H}_B", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & B_0 \arrow[d, equal] \\
A_0 \arrow[r, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow \rho^0_M"] & B_0 \arrow[rr, "\mathcal{H}_B", "\shortmid"{marking}] & & B_0 \arrow[d, equal] & = & A_0 \arrow[rr, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow \rho^0_M"] & & B_0 \arrow[r, "\mathcal{H}_B", "\shortmid"{marking}] & B_0 \arrow[d, equal] \\
A_0 \arrow[rrr, "M_0"', "\shortmid"{marking}] & & & B_0 & & A_0 \arrow[rrr, "M_0"', "\shortmid"{marking}] & & & B_0
\end{tikzcd}$$

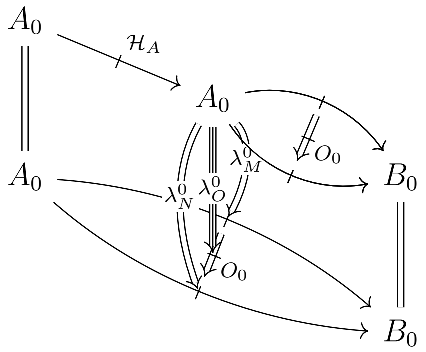
The compatibility law for λ^0_M and ρ^0_M says that

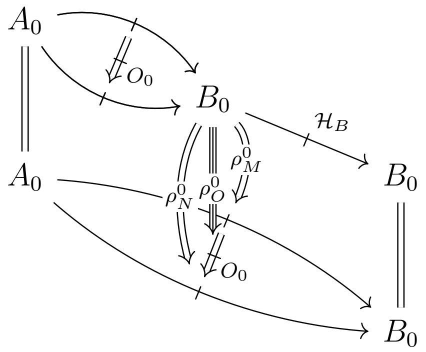
$$\begin{tikzcd}
A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & A_0 \arrow[r, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \rho^0_M"] & B_0 \arrow[r, "\mathcal{H}_B", "\shortmid"{marking}] & B_0 \arrow[d, equal] & & A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow \lambda^0_M"] & A_0 \arrow[r, "M_0", "\shortmid"{marking}] & B_0 \arrow[r, "\mathcal{H}_B", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & B_0 \arrow[d, equal] \\
A_0 \arrow[r, "\mathcal{H}_A", "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow \lambda^0_M"] & A_0 \arrow[rr, "M_0", "\shortmid"{marking}] & & B_0 \arrow[d, equal] & = & A_0 \arrow[rr, "M_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[drrr, phantom, "\Downarrow \rho^0_M"] & & B_0 \arrow[r, "\mathcal{H}_B", "\shortmid"{marking}] & B_0 \arrow[d, equal] \\
A_0 \arrow[rrr, "M_0"', "\shortmid"{marking}] & & & B_0 & & A_0 \arrow[rrr, "M_0"', "\shortmid"{marking}] & & & B_0
\end{tikzcd}$$

Example 3.4.2.14. A $(2,0)$-module in $\mathbb{X}$ consists of a 2-type in $\mathbb{X}$

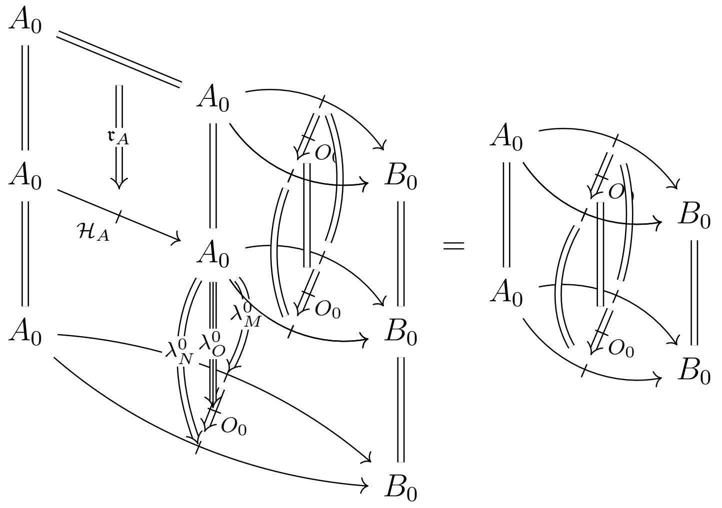
$$\begin{tikzcd} A_0 \arrow[r, bend left=50, "M_0"{name=U}, "\shortmid"{marking}] \arrow[r, bend right=50, "N_0"'{name=D}, "\shortmid"{marking}] & B_0 \arrow[from=U, to=D, Rightarrow, "O_0"] \end{tikzcd}$$

such that M_0 and N_0 underlie $(1,0)$-modules, together with 2-terms

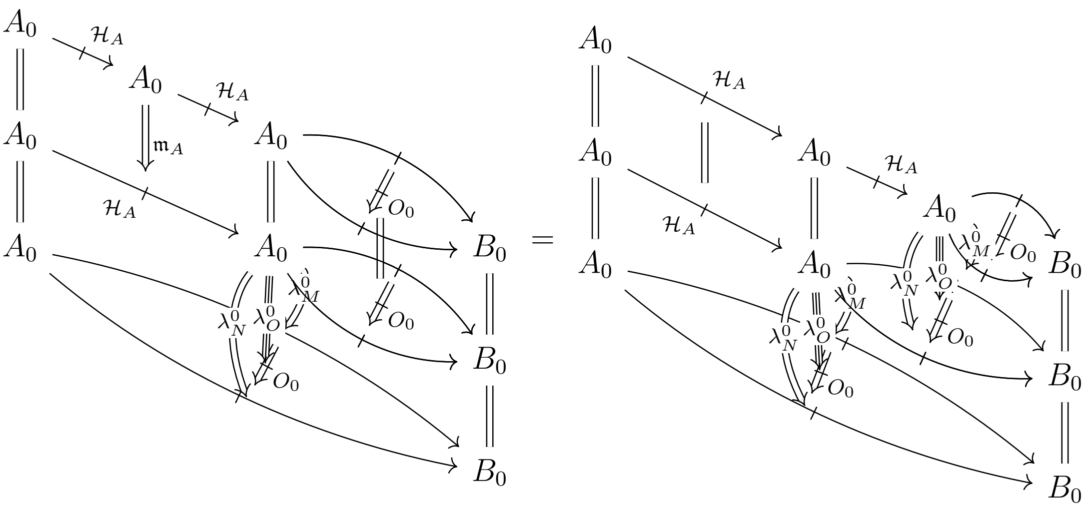




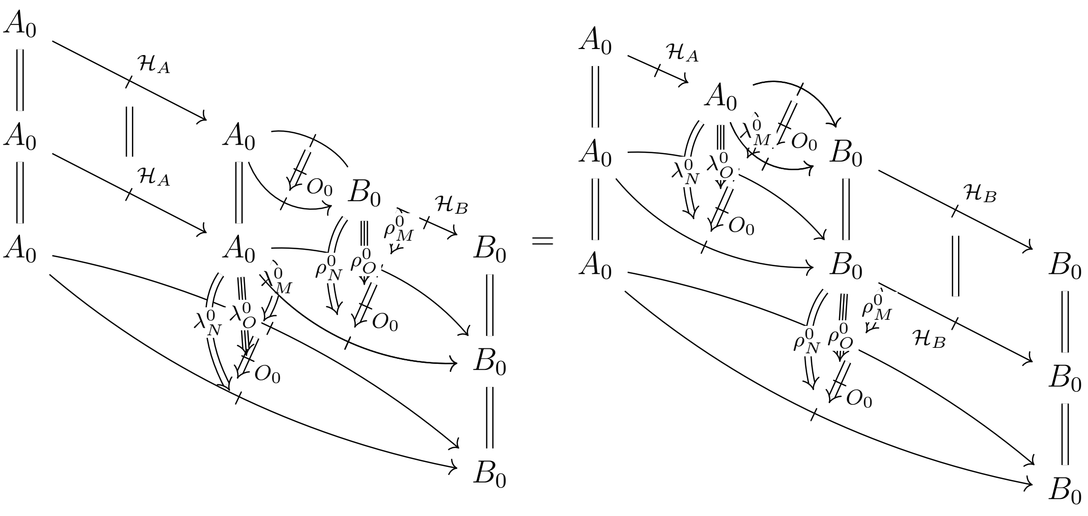
The laws for the action λ^0_M say that



and



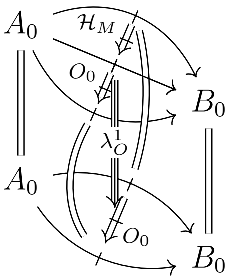
The laws for the action ρ^0_M are similar. The compatibility law says that

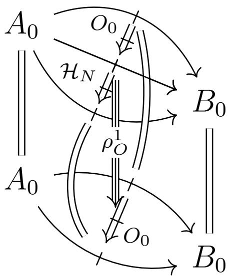


Example 3.4.2.15. A $(2,1)$-module in $\mathbb{X}$ consists of a 2-type

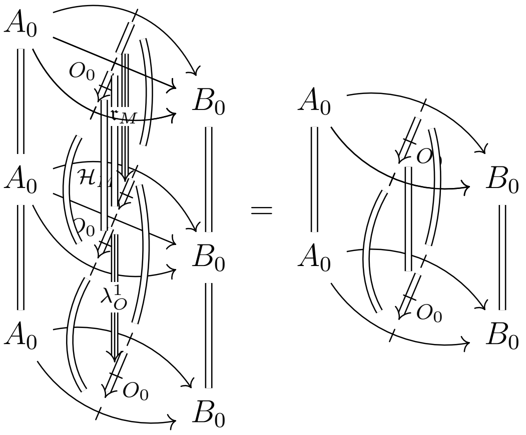
$$\begin{tikzcd}
A_0 \arrow[r, bend left, "M_0"] \arrow[r, bend right, "N_0"'] \arrow[r, phantom, "\Downarrow O_0"] & B_0
\end{tikzcd}$$

such that M_0 and N_0 underlie 1-monoids, together with 2-terms

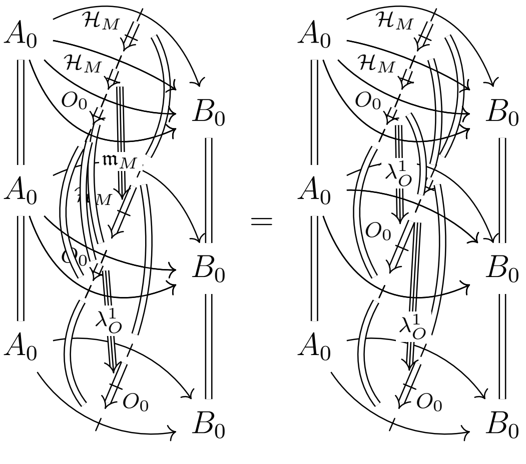




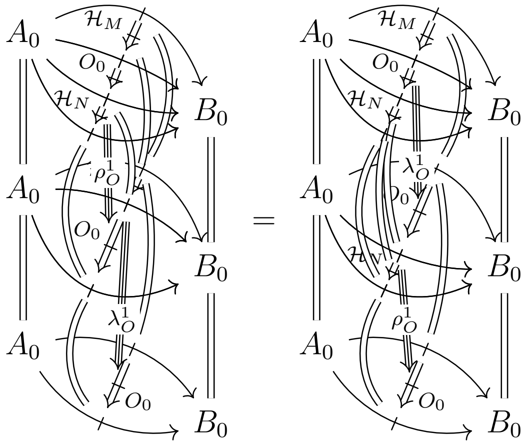
The laws for the action λ^1_O say that



and



The laws for the action ρ^1_O are similar. The compatibility law of λ^1_O and ρ^1_O says that



Example 3.4.2.16. Every l-monoid induces a canonical $(l+1, l)$-module over itself: given an l-monoid A, the $(l+1)$-type $\mathcal{H}_A$ can be made into an $(l+1, l)$-module by defining $\lambda_{\mathcal{H}_A} = \rho_{\mathcal{H}_A} = \mathfrak{m}_A$.

Definition 3.4.2.17. An n-type of $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ is an (n, l)-module in $\mathbb{X}$.

Suppose that Γ is an n-context in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. Then, the action terms of the modules in Γ assemble into *action substitutions* on Γ. Suppose that $x : A$ is an l-variable in Γ. When $x \notin t_l\Gamma$, we define the π-shaped n-substitution λ^Γ_x so that, for each $y : M$ in Γ_0,

$$(\lambda^\Gamma_x)_y = \begin{cases} \lambda_M & \text{if } \dim y > l \text{ and } x = s_l y \\ \mathrm{id}_M & \text{otherwise} \end{cases}$$

By Remark 3.1.2.1, we have that $\lambda^\Gamma_x : \Gamma_0 \oplus_x \mathcal{H}_A \to \Gamma_0$. Similarly, when $x \notin s_l\Gamma$, we define the substitution $\rho^\Gamma_x : \Gamma_0 \oplus_x \mathcal{H}_A$ so that, for each $y : M$ in Γ_0,

$$(\rho^\Gamma_x)_y = \begin{cases} \rho_M & \text{if } \dim y > l \text{ and } x = t_l y \\ \mathrm{id}_M & \text{otherwise} \end{cases}$$

More generally, suppose that $S = \{x_1, \ldots, x_k\} \subseteq \Gamma(l)$ is a set of l-variables. Let $\Gamma \oplus_x \mathcal{H} = \Gamma_0 \oplus_{x_1} \mathcal{H}_{A_1} \oplus_{x_2} \cdots \oplus_{x_k} \mathcal{H}_{A_k}$. Then, when $S \cap t_l\Gamma(l) = \emptyset$, we define $\lambda^\Gamma_S : \Gamma_0 \oplus_S \mathcal{H} \to \Gamma_0$ so that, for each $y : M \in \Gamma$,

$$(\lambda^\Gamma_S)_y = \begin{cases} \lambda_M & \text{if } \dim y > l \text{ and } s_l y \in S \\ \mathrm{id}_M & \text{otherwise} \end{cases}$$

When $S \cap s_l\Gamma(l) = \emptyset$, we define $\rho^\Gamma_S : \Gamma_0 \oplus_S \mathcal{H} \to \Gamma_0$ so that, for each $y : M \in \Gamma$,

$$(\rho^\Gamma_S)_y = \begin{cases} \rho_M & \text{if } \dim y > l \text{ and } t_l y \in S \\ \mathrm{id}_M & \text{otherwise} \end{cases}$$

Example 3.4.2.18. Suppose that Γ is the following 1-context in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$:

$$\begin{tikzcd}
A \arrow[r, "M", "\shortmid" marking] & B \arrow[r, "N", "\shortmid" marking] & C \arrow[r, "O", "\shortmid" marking] & D
\end{tikzcd}$$

Let $x : B$ be the unique variable with type B in Γ. Then λ^Γ_x is the substitution:

$$\begin{tikzcd}
A \arrow[r, "M", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & B \arrow[r, "\mathcal{H}_B", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "N", "\shortmid" marking] \arrow[d, Rightarrow, "\lambda^0_N"] & C \arrow[r, "O", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & D \arrow[d, equal] \\
A \arrow[r, "M"', "\shortmid" marking] & B \arrow[rr, "N"', "\shortmid" marking] & & C \arrow[r, "O"', "\shortmid" marking] & D
\end{tikzcd}$$

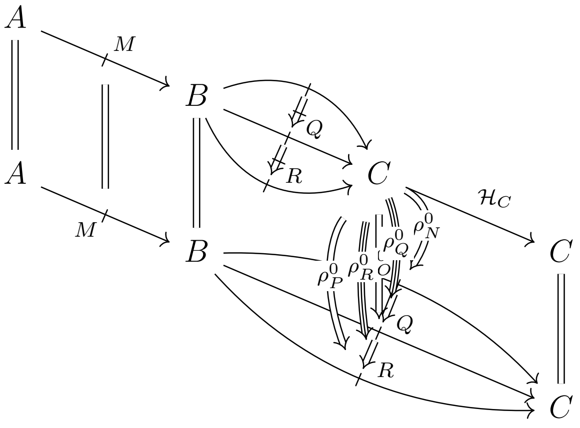
On the other hand, ρ^Γ_x is the following substitution

$$\begin{tikzcd}
A \arrow[r, "M", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "\mathcal{H}_B", "\shortmid" marking] \arrow[d, Rightarrow, "\rho^0_M"'] & B \arrow[r, "N", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & C \arrow[r, "O", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Vert"] & D \arrow[d, equal] \\
A \arrow[rr, "M"', "\shortmid" marking] & & B \arrow[r, "N"', "\shortmid" marking] & C \arrow[r, "O"', "\shortmid" marking] & D
\end{tikzcd}$$

Example 3.4.2.19. Suppose that Γ is the following 2-context in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$:

$$\begin{tikzcd}
A \arrow[r, "M", "\shortmid" marking] & B \arrow[r, bend left=50, "N", "\shortmid" marking] \arrow[r, "O" description] \arrow[r, bend right=50, "P"', "\shortmid" marking] \arrow[r, phantom, bend left=25, "\Downarrow Q"] \arrow[r, phantom, bend right=25, "\Downarrow R"] & C
\end{tikzcd}$$

Suppose that x is the unique 0-variable with type C in Γ. Then λ^Γ_x is undefined, and ρ^Γ_x is the following substitution:



Suppose that Γ and M are an n-context and an n-type in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ respectively. An (n,l)-*module prehomomorphism* $f : \Gamma \to M$ *in* $\mathbb{X}$ is a term $f_0\Gamma_0 \to M_0$ together with term-wise parallel l-monoid homomorphisms $s_l f$ and $t_l f$ such that $(s_l f)_0 = s_l f_0$ and $(t_l f)_0 = t_l f_0$. Given a prehomomorphism $f : \Gamma \to M$ and an l-variable $x \in \Gamma(l)$, there are always two canonical ways to form an n-term $\Gamma_0 \oplus_x \mathcal{H}_A \to M_0$ in $\mathbb{X}$; we will denote these terms by $\mathfrak{J}^+_x(f)$ and $\mathfrak{J}^-_x(f_0)$. We define these $\mathfrak{J}$-*terms* by case analysis depending on whether x is in the l-source or l-target of Γ:

$$\mathfrak{J}^+_x(f) = \begin{cases} \lambda^\Gamma_x; f_0 & \text{if } x \notin t_l\Gamma \\ (f_0 \odot_l \mathcal{H}_{t_l f, x}); \rho_M & \text{if } x \in t_l\Gamma \end{cases}$$
$$\mathfrak{J}^-_x(f) = \begin{cases} \rho^\Gamma_x; f_0 & \text{if } x \notin s_l\Gamma \\ (\mathcal{H}_{s_l f, x} \odot_l f_0); \lambda_M & \text{if } x \in s_l\Gamma \end{cases}$$

When $n = l + 1$, we have that

$$\mathfrak{J}^+_x(f), \mathfrak{J}^-(f) : f \twoheadrightarrow f$$

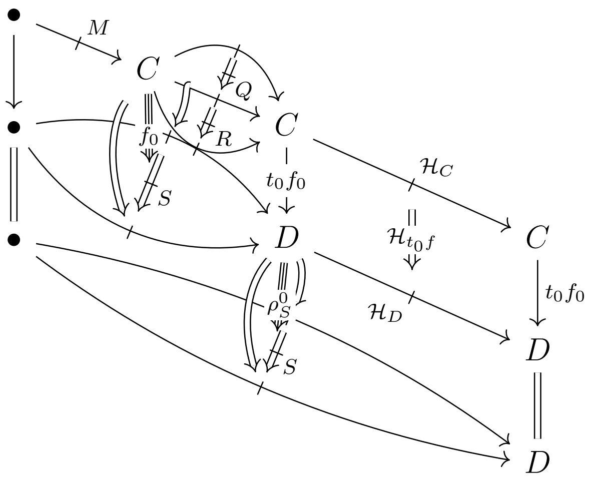
and when $n > l + 1$, we have that

$$\mathfrak{J}^+_x(f) : \mathfrak{J}^+_x(sf) \twoheadrightarrow \mathfrak{J}^+_x(tf), \qquad \mathfrak{J}^-_x(f) : \mathfrak{J}^-_x(sf) \twoheadrightarrow \mathfrak{J}^-_x(tf).$$

Remark 3.4.2.20. Slightly weaker conditions suffice for the construction of $\mathfrak{J}^+_x(f)$ and $\mathfrak{J}^-_x(f)$. For example, if $x \in t_l\Gamma$, then we need only require that $t_l f_0$ underlies a term $t_l f$ in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ and that M_0 underlies a term M in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$.

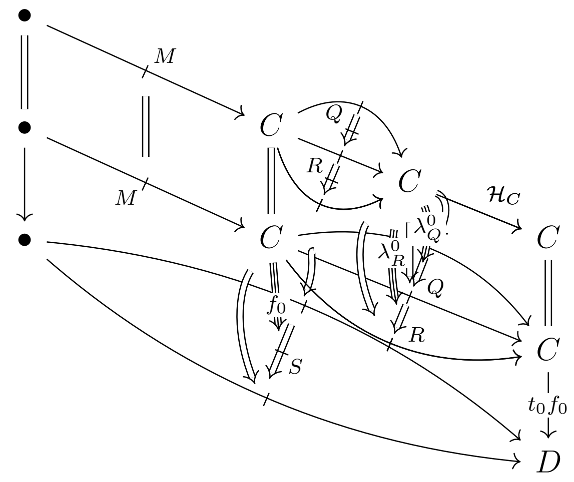
Example 3.4.2.21. Suppose that Γ is the 2-context in $\mathrm{Mod}^{\{0\}}\,\mathbb{X}$ defined in Example 3.4.2.19. Suppose that $x : C$ is the unique 0-variable in Γ with type C. Suppose that $f : \Gamma \to S$ is a $(2, 0)$-module prehomomorphism. Then,

$$\mathfrak{J}^+_x(f) \quad =$$



and

$$\mathfrak{J}^-_x(f) \quad =$$



Remark 3.4.2.22. By construction, we always have that

$$\mathfrak{r}_x; \mathfrak{J}^+_x(f) = f = \mathfrak{r}_x; \mathfrak{J}^-_x(f).$$

We will soon see that $\mathfrak{J}^+_x$ and $\mathfrak{J}^-_x$ agree in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. Furthermore, in this case, the map $\mathfrak{J}^+_x = \mathfrak{J}^-_x$ is exactly the inverse to composition with $\mathfrak{r}_x$. See Proposition 3.4.2.38.

More generally, suppose that $f : \Gamma \to M$ is an (n, l)-module prehomomorphism, and let $S = \{x_1, \ldots, x_m\}$ be a set of l-variables in Γ. Then, we define

$$\mathfrak{J}^+_S(f) = \mathfrak{J}^+_{x_1} \cdots \mathfrak{J}^+_{x_m}(f), \qquad \mathfrak{J}^-_S(f) = \mathfrak{J}^-_{x_1} \cdots \mathfrak{J}^-_{x_m}(f).$$

The right-hand expressions always make sense because the weak conditions mentioned in Remark 3.4.2.20 are always satisfied. The following proposition tells us that the order of the x_i makes no difference, and so $\mathfrak{J}^+_S(f)$ and $\mathfrak{J}^-_S(f)$ are well-defined.

Lemma 3.4.2.23. *Suppose that $f : \Gamma \to M$ is an (n, l)-module prehomomorphism. Let $x : A$ and $y : B$ be distinct l-variables in Γ. Then,*

$$\mathfrak{J}^+_x \mathfrak{J}^+_y(f) = \mathfrak{J}^+_y \mathfrak{J}^+_x(f)$$

Proof. We will prove the statement for $\mathfrak{J}^+$. The statement for $\mathfrak{J}^-$ follows by a symmetrical argument. First suppose that $x \notin t_l\Gamma$, and $y \notin t_l\Gamma$. Then, since $x \neq y$, we have that $\rho_x^{\Gamma \oplus_y \mathcal{H}_B}; \rho_y^{\Gamma} = \rho_y^{\Gamma \oplus_x \mathcal{H}_A}; \rho_x^{\Gamma}$, and so

$$\mathfrak{J}^+_x \mathfrak{J}^+_y(f) = \rho_x^{\Gamma \oplus_y \mathcal{H}_B}; \rho_y^{\Gamma}; f = \rho_y^{\Gamma \oplus_x \mathcal{H}_A}; \rho_x^{\Gamma}; f = \mathfrak{J}^+_y \mathfrak{J}^+_x(f).$$

Now suppose that $x \notin t_l\Gamma$, and $y \in t_l\Gamma$. Then,

$$\mathfrak{J}^+_x \mathfrak{J}^+_y(f) = \rho_x^{\Gamma \oplus \mathcal{H}_y}; (f \odot_l \mathcal{H}_{t_l f, y}); \rho_M = \mathfrak{J}^+_y(\rho_x^{\Gamma}; f) = \mathfrak{J}^+_y \mathfrak{J}^+_x(f).$$

A similar argument holds when $x \in t_l\Gamma$, and $y \notin t_l\Gamma$. Finally, suppose that $x \notin t_l\Gamma$ and $y \notin t_l\Gamma$. Then, we have that

$$\begin{aligned}\mathfrak{I}^+_x\mathfrak{I}^+_y(f) &= \mathfrak{I}^+_x((f \odot_l \mathcal{H}_{t_lf,y}); \rho_M)\\ &= ((f \odot_l \mathcal{H}_{t_lf,y}); \rho_M) \odot_l \mathcal{H}_{t_lf,x}); \rho_M\\ &= (f \odot_l \mathcal{H}_{t_lf,y} \odot_l \mathcal{H}_{t_lf,x}); (\rho_M \odot_l \mathrm{id}_{t_lM}); \rho_M\\ &= (f \odot_l \mathcal{H}_{t_lf,y} \odot_l \mathcal{H}_{t_lf,x}); (\rho_M \odot_l \mathrm{id}_{t_lM}); \rho_M\end{aligned}$$

since ρ_M respects $\mathfrak{m}_{t_lM}$, we have that

$$\begin{aligned}\mathfrak{I}^+_x\mathfrak{I}^+_y(f) &= (f \odot_l \mathcal{H}_{t_lf,y} \odot_l \mathcal{H}_{t_lf,x}); (\rho_M \odot_l \mathrm{id}_{t_lM}); \rho_M\\ &= (f \odot_l \mathcal{H}_{t_lf,y} \odot_l \mathcal{H}_{t_lf,x}); (\mathrm{id}_{t_lM} \odot_l \mathfrak{m}_{t_lM}); \rho_M\\ &= f \odot_l ((\mathcal{H}_{t_lf,y} \odot_l \mathcal{H}_{t_lf,x}); \mathfrak{m}_{t_lM}); \rho_M\\ &= (f \odot_l \mathcal{H}_{t_lf,\{x,y\}}); \rho_M\end{aligned}$$

The last equality follows from Lemma 3.4.2.11. By a symmetrical argument, we have that

$$\mathfrak{I}^+_y\mathfrak{I}^+_x(f) = (f \odot_l \mathcal{H}_{t_lf,\{x,y\}}); \rho_M.$$

Hence, we have that $\mathfrak{I}^+_x\mathfrak{I}^+_y(f) = \mathfrak{I}^+_y\mathfrak{I}^+_x(f)$ as required. □

Proposition 3.4.2.24. *Suppose that $f : \Gamma \to M$ is an (n,l)-module prehomomorphism. The following properties immediately follow from the definition of $\mathfrak{I}^+_S$ and $\mathfrak{I}^-_S$ and the proof of the preceding lemma:*

- *For any l-variable $x \in \Gamma(l)$, we have that*

$$\mathfrak{I}^+_{\{x\}}(f) = \mathfrak{I}^+_x(f).$$

- *When $S \subseteq (s_l\Gamma)(l)$, we have that*

$$\mathfrak{I}^-_x(f) = (\mathcal{H}_{t_lf,S} \odot_l f_0); \lambda_M.$$

- *When $S \subseteq (t_l\Gamma)(l)$, we have that*

$$\mathfrak{I}^+_S(f) = (f_0 \odot_l \mathcal{H}_{t_lf,S}); \rho_M.$$

- *When $S, T \subseteq \Gamma(l)$ are disjoint, we have that*

$$\mathfrak{I}^+_S\mathfrak{I}^+_T(f) = \mathfrak{I}^+_{S\cup T}(f) = \mathfrak{I}^+_T\mathfrak{I}^+_S(f)$$
$$\mathfrak{I}^-_S\mathfrak{I}^-_T(f) = \mathfrak{I}^-_{S\cup T}(f) = \mathfrak{I}^-_T\mathfrak{I}^-_S(f)$$

- *By the unit laws we have that*

$$\mathfrak{r}_S; \mathfrak{J}^+(f) = f = \mathfrak{r}_S \mathfrak{J}^-_S(f).$$

Definition 3.4.2.25. An (n, l)*-module homomorphism* $f : \Gamma \to M$ *in* $\mathbb{X}$ is an (n, l)-module prehomomorphism satisfying the *equivariance law*,

$$\mathfrak{J}^+_x(f) = \mathfrak{J}^-_x(f),$$

for each l-variable $x \in \Gamma$. It follows immediately from this definition that:

- The source and target of an $(l + 1, l)$-module homomorphism are l-monoid homomorphisms.
- When $n > l + 1$, the source and target of an (n, l)-module homomorphism are $(n - 1, l)$-module homomorphisms.

Remark 3.4.2.26. Suppose that $f : \Gamma \to M$ is an (n, l)-module homomorphism. Suppose that $S \subseteq \Gamma(l)$ is a set of l-variables. Then, it follows immediately that

$$\mathfrak{J}^+_S(f) = \mathfrak{J}^-_S(f).$$

Definition 3.4.2.27. When $n < l$, an n-term in $\mathrm{Mod}^{\{l\}}\, \mathbb{X}$ is an (n, l)-module homomorphism in $\mathbb{X}$.

Suppose that $f : \Gamma \to \Delta$ is an n-substitution in $\mathrm{Mod}^{\{l\}}\, \mathbb{X}$. Let $x : A$ be an l-variable in Γ. Then the the actions of A on the types in Γ allow us to construct a number of different substitutions $\Gamma \oplus_x \mathcal{H}_A \to \Delta$. For example, suppose that f is the following 1-substitution:

$$\begin{tikzcd}
\bullet \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow f_1"] & A \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\Downarrow f_2"] & A \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow f_3"] & \bullet \arrow[d] \\
\bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet
\end{tikzcd}$$

Suppose that x is the 0-variable with type A whose type is labeled in this diagram. Then the terms $\mathfrak{J}^+_x(f_1)$, $\mathfrak{J}^-_x(f_2)$, $\mathfrak{J}^+_x(f_2)$ and $\mathfrak{J}^-_x(f_3)$ all induce a distinct substitution $\Gamma \oplus_x \mathcal{H}_A \to \Delta$. In general, for each $y \in \Delta$ such that $x \in \Gamma_y(l)$, we have $\mathfrak{J}$-terms $\mathfrak{J}^+_x(f_y)$ and $\mathfrak{J}^-_x(f_y)$. The following result will enable us to describe the substitutions that can be built from these $\mathfrak{J}$-terms systematically.

Definition 3.4.2.28. Let $f : \Gamma \to \Delta$ be a π-shaped n-substitution in a globular multicategory $\mathbb{X}$. Let x be an l-variable in Γ. Let $\pi|_{f\{x\}}$ be the sub-globular set of π such that for each k-variable $y \in \pi(k)$, if $k \geq l$, then

$$y \in \pi|_{f\{x\}}(k) \iff x \in \Gamma_y(l),$$

and if $k < l$, then $y \in \pi|_{f\{x\}}(k)$ if and only if there exists some l-variable $z \in \pi|_{f\{x\}}(l)$ such that $s_k z = y$ or $t_k z = y$.

Lemma 3.4.2.29. *The globular set $\pi|_{f\{x\}}$ is an $(l-1)$-trivial pasting diagram.*

Proof. First suppose that π is $(l-1)$-trivial, and that

$$\pi = \pi_1 \odot_l \cdots \odot_l \pi_m$$

where each π_i is l-trivial, and $m \geq 0$. Suppose that $m = 0$. Then, $\pi = \mathbf{D}^l$. Since $x \in \Gamma(l)$, it follows that $\pi|_{f\{x\}} = \mathbf{D}^l$. This is certainly an $(l-1)$-trivial pasting diagram. Hence, suppose that $m > 0$. Then each π_i corresponds to a π_i-shaped substitution $f_i : \Gamma_i \to \Delta_i$ such that

$$f = f_1 \odot_l \cdots \odot_l f_m,$$

Thus, we have that

$$\Gamma = \Gamma_1 \odot_l \cdots \odot_l \Gamma_m$$

We claim that $S = \{1 \leq i \leq m \mid x \in \Gamma_i(l)\}$ is a sequence $\{j, j+1, \cdots, j+k\}$ for some j and k. Hence, suppose that $x \in \Gamma_i$. Let ρ_i be the shape of Γ_i. First, suppose that $\rho_i = \mathbf{D}^l$. Then $x \in s_l\Gamma_i$ and $x \in t_l\Gamma_i$. It follows that if $i > 0$, then $x \in t_l\Gamma_{i-1}$, and if $i < l$, then $x \in s_l\Gamma_{i+1}$. Now suppose that $\pi_i \neq \mathbf{D}^l$. If $x \in s_l\Gamma_i$, then $x \in t_l\Gamma_{i-1}$ when $i > 0$. Furthermore, in this case $x \notin \Gamma_j$ for any $j > i$. Similarly, if $x \in t_l\Gamma_i$, then $x \in s_l\Gamma_{i+1}$ when $i < l$. Furthermore, in this case $x \notin \Gamma_j$, for any $j < i$. Finally, suppose that $x \notin s_l\Gamma_i$ and $x \notin t_l\Gamma_i$. Then, $x \notin \Gamma_j$ for any $j \neq i$. Combining these observations, we find that S must be a sequence $\{j, j+1, \ldots, j+k\}$. Consequently, we have that

$$\pi|_{f\{x\}} = \pi_j \odot_l \pi_{j+1} \odot_l \cdots \pi_{j+k}.$$

This is an $(l-1)$-trivial pasting diagram.

We now prove the claim for $(l-k-1)$-trivial pasting diagrams by induction on $0 \leq k \leq l$. We have just proved the base case, when $k = 0$. Hence, suppose that $k > 0$, and that

$$\pi = \pi_1 \odot_{l-k} \cdots \odot_{l-k} \pi_m$$

where each π_i is $(l-k)$-trivial. Then, there must be a unique i such that $x \in \Gamma_i$. Consequently, we have that

$$\pi|_{f\{x\}} = \pi_i|_{f\{x\}}.$$

However, this is an $(l-1)$-trivial pasting diagram by the inductive hypothesis. This completes the induction. Since every pasting diagram is (-1)-trivial, we have proved the claim. $\square$

Now suppose that $f : \Gamma \to \Delta$ is a π-shaped n-substitution in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$, and that $x : A$ is an l-variable in Γ. Suppose that $\Delta|_{f\{x\}}$ is the context of Δ induced by $\pi|_{f\{x\}}$. By the lemma above, we have that

$$\Delta|_{f\{x\}} = \Delta_1 \odot_l \Delta_2 \odot_l \cdots \odot_l \Delta_m$$

for some l-trivial Δ_i and some $m \geq 0$. For each $1 \leq i \leq m$, we define substitutions $\mathfrak{J}^+_{x,i}(f)$, $\mathfrak{J}^-_{x,i}(f) : \Gamma_0 \oplus_x \mathcal{H}_A \to \Delta$ so that, for each $y \in \Delta$,

$$(\mathfrak{J}^+_{x,i}(f))_y = \begin{cases} \mathfrak{J}^+_x(f_y) & \text{if } \dim y > l \text{ and } t_l y = t_l \Delta_i \\ f_y & \text{otherwise} \end{cases}$$

$$(\mathfrak{J}^-_{x,i}(f))_y = \begin{cases} f_y & \text{if } \dim y \leq l \\ \mathfrak{J}^-_x(f_y) & \text{if } \dim y > l \text{ and } s_l y = s_l \Delta_i \\ f_y & \text{otherwise} \end{cases}$$

Note that $t_k y = t_k \Delta_i$ if and only if $y \in \Delta_i$ if and only if $s_k y = s_k \Delta_i$. Hence, since each term in f satisfies the equivariance laws, we obtain:

Lemma 3.4.2.30. *For each i, we have that*

$$\mathfrak{J}^+_{x,i}(f) = \mathfrak{J}^-_{x,i}(f).$$

Example 3.4.2.31. Suppose again that $f : \Gamma \to \Delta$ is the following 1-substitution:

$$\begin{tikzcd}
\bullet \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow f_1"] & A \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\Downarrow f_2"] & A \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\Downarrow f_3"] & \bullet \arrow[d] \\
\bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet \arrow[r, "|" marking] & \bullet
\end{tikzcd}$$

Suppose that $x : A$ is the 0-variable in Γ whose type A is labelled in this diagram. Then,

$$\begin{aligned}
\mathfrak{J}^+_{x,1} &= \mathfrak{J}^+_x(f_1) \odot_0 f_2 \odot_0 f_3, & \mathfrak{J}^-_{x,1} &= \mathfrak{J}^+_x(f_1) \odot_0 f_2 \odot_0 f_3, \\
\mathfrak{J}^+_{x,2} &= f_1 \odot_0 \mathfrak{J}^+_x(f_2) \odot_0 f_3, & \mathfrak{J}^-_{x,2} &= f_1 \odot_0 \mathfrak{J}^-_x(f_2) \odot_0 f_3, \\
\mathfrak{J}^+_{x,3} &= f_1 \odot_0 f_2 \odot_0 \mathfrak{J}^+_x(f_3), & \mathfrak{J}^-_{x,3} &= f_1 \odot_0 f_2 \odot_0 \mathfrak{J}^-_x(f_3).
\end{aligned}$$

Suppose that $f : \Gamma \to \Delta$ and $g : \Delta \to M$ are a composable pair $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. We define the module prehomomorphism $f;g$ by

$$(f;g)_0 = f_0; g_0$$

Our goal now is to show that $f;g$ is a module homomorphism; that is, $f;g$ satisfies the required equivariance laws. Suppose that x is an l-variable in Γ, and write

$$\Delta|_{f\{x\}} = \Delta_1 \odot_l \cdots \odot_l \Delta_m$$

where each Δ_i is l-trivial. We will prove that $f;g$ satisfies the required equivariance laws by induction on m.

Proposition 3.4.2.32. *When* $m = 0$*, we have that* $\mathfrak{J}^+_x(f;g) = \mathfrak{J}^-_x(f;g)$*.*

Proof. First, suppose that $m = 0$. Then $f = [f']$ for some l-term f'. Consequently, $x \in s_l\Gamma$, and $x \in t_l\Gamma$, and $\Delta = s_l\Delta = t_l\Delta$. Hence, by Proposition 3.4.2.24, we have that

$$\begin{aligned}
\mathfrak{J}^+_x(f;g) &= ((f'_0;g_0) \odot_l \mathcal{H}_{t_l(f;g),x}); \rho_M \\
&= ((f'_0;g_0) \odot_l (\mathcal{H}_{f',x}; \mathcal{H}_{t_l g}); \rho_M \\
&= \mathcal{H}_{f',x}; (g_0 \odot_l \mathcal{H}_{t_l g}); \rho_M \\
&= \mathcal{H}_{f',x}; \mathfrak{J}^+_{t_l\Delta(l)}(g) \\
&= \mathcal{H}_{f',x}; \mathfrak{J}^-_{t_l\Delta(l)}(g) \\
&= \mathcal{H}_{f',x}; (\mathcal{H}_{s_l g} \odot_l g_0); \lambda_M \\
&= ((\mathcal{H}_{f',x}; \mathcal{H}_{s_l g}) \odot_l (f'_0; g_0)); \lambda_M \\
&= (\mathcal{H}_{s_l(f;g),x} \odot_l ((f'_0; g_0)); \lambda_M \\
&= \mathfrak{J}^-_x(f;g).
\end{aligned}$$

□

Now suppose that $m > 0$. Our approach will be as follows:

- Firstly, we will show that

  $$J^+_x(f;g) = \mathfrak{J}^+_{x,m}(f); g_0,$$

  and that

  $$\mathfrak{J}^-_x(f;g) = \mathfrak{J}^-_{x,1}(f); g_0.$$

  See Lemma 3.4.2.33.

- Secondly, we will show that

$$\mathfrak{I}^{+}_{x,m}(f); g_0 = \mathfrak{I}^{-}_{x,1}(f); g_0.$$

  See Corollary 3.4.2.36.

Lemma 3.4.2.33. *We have that*

$$J^{+}_{x}(f; g) = \mathfrak{I}^{+}_{x,m}(f); g_0,$$

and

$$\mathfrak{I}^{-}_{x}(f; g) = \mathfrak{I}^{-}_{x,1}(f); g_0.$$

Proof. For any $1 \leq i \leq m$, we have that $t_l\Delta_i \in t_l\Delta$ implies $i = m$. The converse holds if and only if $x \in t_l\Gamma$. Hence,

$$\mathfrak{I}^{+}_{x,m}(f) = \begin{cases} (f_0 \oplus_{t_l\Delta_m} \mathcal{H}_{t_l f_{\Delta_m},x}); \rho^{\Delta}_{\Delta_m(l)} & \text{if } x \in t_l\Gamma \\ \lambda^{\Gamma}_{x}; f_0 & \text{if } x \notin t_l\Gamma \end{cases}$$

By a similar argument,

$$\mathfrak{I}^{-}_{x,1}(f) = \begin{cases} (\mathcal{H}_{s_l f,x} \oplus_{s_l\Delta_1} \mathcal{H}_{s_l f_{\Delta_1},x}); \lambda^{\Delta}_{\Delta_1(l)} & \text{if } x \in s_l\Gamma \\ \rho^{\Gamma}_{x}; f_0 & \text{if } x \notin s_l\Gamma \end{cases}$$

It follows immediately that when $x \notin t_l\Gamma$, we have that

$$J^{+}_{x}(f; g) = \mathfrak{I}^{+}_{x,m}(f); g_0,$$

and when $x \notin s_l\Gamma$, we have that

$$\mathfrak{I}^{-}_{x}(f; g) = \mathfrak{I}^{-}_{x,1}(f); g_0.$$

On the other hand, when $x \in t_l\Gamma$, we have that

$$\begin{aligned} \mathfrak{I}^{+}_{x,m}(f); g_0 &= (f_0 \oplus_{t_l\Delta_l} \mathcal{H}_{t_l f_{\Delta_m},x}); \rho^{\Delta}_{t_l\Delta_m}; g_0 \\ &= (f_0 \oplus_{t_l\Delta_l} \mathcal{H}_{t_l f_{\Delta_m},x}); \mathfrak{I}^{-}_{t_l\Delta_m}(g). \end{aligned}$$

Similarly, when $x \notin s_l\Gamma$, we have that

$$\begin{aligned} \mathfrak{I}^{-}_{x,1}(f); g_0 &= (f_0 \oplus_{s_l\Delta_1} \mathcal{H}_{s_l f_{\Delta_1},x}); \lambda^{\Delta}_{s_l\Delta_1}; g_0 \\ &= (f_0 \oplus_{s_l\Delta_1} \mathcal{H}_{s_l f_{\Delta_1},x}); \mathfrak{I}^{+}_{s_l\Delta_1}(g). \end{aligned}$$

The result now follows from Lemma 3.4.2.34 below. □

Lemma 3.4.2.34. *Suppose that $f : \Gamma \to \Delta$ and $g : \Delta \to M$ are terms in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. Suppose that x is an l-variable $x \in t_l\Gamma$, such that $\Delta|_x = \Delta_1 \odot_l \cdots \odot_l \Delta_m$, where each Δ_i is i-trivial, and where $m > 0$. Then, we have that*

$$\mathfrak{J}^+_x(f;g) = \ (f_0 \oplus_{t_l\Delta_m} \mathcal{H}_{t_l f_{\Delta_m},x}); \mathfrak{J}^-_{t_l\Delta_m}(g)$$

Similarly, for any l-variable $x \in s_l\Gamma$, we have that

$$\mathfrak{J}^-_x(f;g) = (f_0 \oplus_{s_l\Delta_1} \mathcal{H}_{s_l f_{\Delta_1},x}); \mathfrak{J}^+_{s_l\Delta_1}(g).$$

Proof. Will will prove that the above description of $\mathfrak{J}^+_x(f;g)$ is correct. The description of $\mathfrak{J}^-_x(f;g)$ follows by a symmetrical argument. Suppose that $x \in t_l\Gamma$. Then

$$\begin{aligned}
\mathfrak{J}^+_x(f;g) &= ((f_0;g_0) \odot_l \mathcal{H}_{t_l(f;g),x}); \rho_M \\
&= (f_0 \odot_l \mathcal{H}_{t_l f,x}); (g_0 \odot_l \mathcal{H}_{t_l g}); \rho_M \\
&= (f_0 \odot_l \mathcal{H}_{t_l f,x}); \mathfrak{J}^+_{t_l\Delta}(g) \\
&= (f_0 \odot_l \mathcal{H}_{t_l f,x}); \mathfrak{J}^-_{t_l\Delta}(g)
\end{aligned}$$

Let $z = t_l\Delta_m$. Then, for each variable $y \in \Delta$, we have that

$$(f_0 \odot_l \mathcal{H}_{t_l f,x})_y = \begin{cases} (f_y)_0 \odot_l (\mathfrak{r}^{t_l\Gamma_y}_{\tilde{x}}; \mathcal{H}_{t_l f_y}) & \text{if } t_l y \in t_l\Delta \text{ and } x \in \Gamma_y \\ (f_y)_0 \odot_l (\mathfrak{r}_{t_l\Gamma_y}; \mathcal{H}_{t_l f_y}) & \text{if } t_l y \in t_l\Delta \text{ and } x \notin \Gamma_y \\ (f_y)_0 & \text{if } \dim y \leq l \text{ or } t_l y \notin t_l\Delta \end{cases}$$

$$= \begin{cases} (f_y)_0 \odot_l \mathcal{H}_{t_l f_y,x} & \text{if } t_l y \in t_l\Delta \text{ and } x \in \Gamma_y \\ (f_y)_0 \odot_l (t_l f_y; \mathfrak{r}_{t_l\Delta_y}) & \text{if } t_l y \in t_l\Delta \text{ and } x \notin \Gamma_y \\ (f_y)_0 & \text{if } \dim y \leq l \text{ or } t_l y \notin t_l\Delta \end{cases}$$

$$= \begin{cases} (f_y)_0 \odot_l \mathcal{H}_{f_z,x} & \text{if } t_l y = z \\ (f_y)_0; (\mathrm{id}_y \odot_l \mathfrak{r}_{t_l\Delta_y}) & \text{if } t_l y \in (t_l\Delta)(l) \setminus \{z\} \\ (f_y)_0 & \text{if } \dim y \leq l \text{ or } t_l y \notin t_l\Delta \end{cases}$$

Hence for each $y \in \Delta$ we define

$$f'_y = \begin{cases} (f_y)_0 \odot_l \mathcal{H}_{f_z,x} & \text{if } t_l y = z \\ (f_y)_0 & \text{if } t_l y \in (t_l\Delta)(l) \setminus \{z\} \\ (f_y)_0 & \text{if } \dim y \leq l \text{ or } t_l y \notin t_l\Delta \end{cases}$$

and

$$r_y = \begin{cases} \mathrm{id}_y \odot_l \mathrm{id}_{\mathcal{H}_{t_l y}} & \text{if } t_l y = z \\ \mathrm{id}_y \odot_l \mathfrak{r}_{t_l y} & \text{if } t_l y \in (t_l\Delta)(l) \setminus \{z\} \\ \mathrm{id}_y & \text{if } \dim y \leq l \text{ or } t_l y \notin t_l\Delta \end{cases}$$

Then, by the analogue of Remark 3.1.2.1 for terms, we have that

$$\begin{aligned}(f_0 \odot_l \mathcal{H}_{t_l f,x}) &= \bigodot_{y\in\Delta} f'_y; r_y \\ &= \bigodot_{y\in\Delta} f'_y; \bigodot_{y\in\Delta} r_y \\ &= (f_0 \oplus_z \mathcal{H}_{f_z,x}); \mathfrak{r}^{\Delta\odot_l \mathcal{H}_{t_l\Delta}}_{(t_l\Delta)(l)\setminus z}\end{aligned}$$

However, by Proposition 3.4.2.24, we have that

$$\mathfrak{J}^-_{(t_l\Delta)(l)}(g) = \mathfrak{J}^-_{(t_l\Delta)(l)\setminus z}(\mathfrak{J}^-_z(g))$$

and so

$$\begin{aligned}\mathfrak{J}^+_x(f;g) &= (f_0 \odot_l \mathcal{H}_{t_l f,x}); \mathfrak{J}^-_{t_l\Delta(l)}(g) \\ &= (f_0 \oplus_z \mathcal{H}_{f_z,x}); \mathfrak{r}^{\Delta\odot_l \mathcal{H}_{t_l\Delta_l}}_{(t_l\Delta)(l)\setminus z}; \mathfrak{J}^-_{(t_l\Delta)(l)\setminus z}(\mathfrak{J}^-_z(g)) \\ &= (f_0 \oplus_z \mathcal{H}_{f_z,x}); \mathfrak{J}^-_z(g) \\ &= (f_0 \oplus_{t_l\Delta_m} \mathcal{H}_{t_l f_{\Delta_m},x}); \mathfrak{J}^-_{t_l\Delta_m}(g)\end{aligned}$$

as required. □

Lemma 3.4.2.35. *Suppose that $g : \Delta \to M$ is a term in* $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. *Then, for all $1 \leq i < l$, we have that*

$$\mathfrak{J}^-_{x,i+1}(f); g_0 = \mathfrak{J}^+_{x,i}(f); g_0.$$

Proof. First, note that since $s_l\Delta_{i+1} = t_l\Delta_i$, we have that $x \in s_l\Gamma_{i+1} \cap t_l\Gamma_i$. Let $z = s_l\Delta_{i+1} = t_l\Delta_i$. Then, for each variable $y : A \in \Delta$, it follows that

$$(\mathfrak{J}^-_{x,i+1}(f))_y = \begin{cases}(\mathcal{H}_{s_l f_y,x} \odot_l (f_y)_0); \lambda_{\Delta_y(l)} & \text{if } s_l y = z \\ (f_y)_0 & \text{if } s_l y \neq z\end{cases}$$

$$(\mathfrak{J}^+_{x,i}(f))_y = \begin{cases}((f_y)_0 \odot_l \mathcal{H}_{t_l f_y,x}); \rho_{\Delta_y(l)} & \text{if } t_l y = z \\ (f_y)_0 & \text{if } t_l y \neq z\end{cases}$$

Let

$$\begin{aligned} f'_y &= \begin{cases}\mathcal{H}_{s_l f_y,x} \odot_l (f_y)_0 & \text{if } s_l y = z \\ (f_y)_0 & \text{if } s_l y \neq z\end{cases} \\ &= \begin{cases}\mathcal{H}_{f_z,x} \odot_l f_y & \text{if } s_l y = z \\ f_y & \text{if } s_l y \neq z\end{cases}\end{aligned}$$

and

$$m_y = \begin{cases}\lambda_{\Delta_y(l)} & \text{if } s_l y = z \\ \mathrm{id}_A & \text{if } s_l y \neq z\end{cases}$$

Then, summing over $y \in \Delta$, and applying Remark 3.1.2.1, we find that

$$\begin{aligned}\mathfrak{J}^{-}_{x,i+1}(f) &= \bigodot_{y\in\Delta}(\mathfrak{J}^{-}_{x,i+1}(f))_y \\ &= \bigodot_{y\in\Delta} f'_y; m_y \\ &= \bigodot_{y\in\Delta} f'_y; \bigodot_{y\in\Delta} m_y \\ &= (f_0 \oplus_z \mathcal{H}_{f_z,x}); \lambda^{\Gamma}_z.\end{aligned}$$

By a similar argument, we have that

$$\mathfrak{J}^{+}_{x,i}(f) = (f_0 \oplus_z \mathcal{H}_{f_z,x}); \rho^{\Gamma}_z.$$

It follows that

$$\begin{aligned}J^{-}_{x,i+1}(f); g_0 &= (f_0 \oplus_z \mathcal{H}_{f_z,x}); \lambda^{\Gamma}_z; g_0 \\ &= (f_0 \oplus_z \mathcal{H}_{f_z,x}); J^{-}_z(g_0) \\ &= (f_0 \oplus_z \mathcal{H}_{f_z,x}); J^{+}_z(g_0) \\ &= (f_0 \oplus_z \mathcal{H}_{f_z,x}); \rho^{\Gamma}_z; g_0 \\ &= J^{+}_{x,i}(f); g_0\end{aligned}$$

as required. □

Corollary 3.4.2.36. *We have that*

$$\mathfrak{J}^{+}_{x,m}(f); g_0 = \mathfrak{J}^{-}_{x,1}(f); g_0$$

Proof. Repeatedly apply Lemma 3.4.2.35 and Lemma 3.4.2.30. □

We can now finally conclude that n-terms in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ can be composed.

Proposition 3.4.2.37. *A composite of (n, l)-module homomorphisms is an (n, l)-module homomorphism.*

In other words, we have proved that $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ is a globular multicategory. We now describe the homomorphism types of l-types in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. Given an l-type M in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$, the homomorphism type of M in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ is defined to be the $(l + 1, l)$-module $\mathcal{H}_M$. The reflexivity term at M is $\mathfrak{r}_M$. The unit and associativity laws ensure that $\mathfrak{r}_M$ is an $(l + 1)$-term in $\mathrm{Mod}^{\{l\}}(\mathbb{X})$. We define $\mathfrak{J}_x$ by

$$\mathfrak{J}_x(f) = \mathfrak{J}^{+}_x(f) = \mathfrak{J}^{-}_x(f),$$

Composition with $\mathfrak{r}_x$ is a bijection with inverse $\mathfrak{J}_x$ by the following proposition:

Proposition 3.4.2.38. *Suppose that* Γ *is a context in* $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$*, and that* $x : A$ *is an* l*-variable in* Γ*. Suppose that* $f : \Gamma \to M$ *and* $g : \Gamma \oplus_x \mathcal{H}_A \to M$ *are* (n, l)*-module homomorphisms in* $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$*. Then, we have that*

$$\mathfrak{r}_x^{\Gamma}; \mathfrak{J}_x^{+}(f) = f,$$
$$\mathfrak{J}_x^{+}(\mathfrak{r}_x^{\Gamma}; g) = g,$$

Proof. From the definition of $\mathfrak{J}_x^{+}(f)$, we have that

$$\mathfrak{r}_x^{\Gamma}; \mathfrak{J}_x^{+}(f) = f$$

and so it suffices to prove the second identity. Suppose that $(\Gamma \oplus_x \mathcal{H}_A)|_{g\{x\}} = \Gamma_1 \odot_l \cdots \odot_l \Gamma_m$, for some l-trivial Γ_i, and $m \geq 0$. Since $\mathcal{H}_A$ is an $(l+1)$-type in $(\Gamma \oplus_x \mathcal{H}_A)|_{g\{x\}}$, we have that $m > 0$. Recall that we denote the added variable in $\Gamma \oplus_x \mathcal{H}_A$ by $\mathcal{H}_x : \mathcal{H}_A$, the source of $\mathcal{H}_x$ by $x_0 : A$, and the target of $\mathcal{H}_x$ by $x_1 : A$. Since $\mathcal{H}_x : \mathcal{H}_A$ is the unique variable in $\Gamma \oplus_x \mathcal{H}_A$ with $t_l \mathcal{H}_x = x_1$, and $\lambda_{\mathcal{H}_A} = \mathfrak{m}_A$, we have that

$$\mathfrak{J}_{x_1}^{-}(g) = \lambda_{x_1}^{\Gamma \oplus_x \mathcal{H}_A}; g = \mathfrak{m}_{\mathcal{H}_x}^{\Gamma \oplus_x \mathcal{H}_A}; g.$$

Similarly, since $\mathcal{H}_x$ is the unique variable in $\Gamma \oplus_x \mathcal{H}_A$ with $s_l \mathcal{H}_x = x_0$, and $\rho_{\mathcal{H}_A} = \mathfrak{m}_A$, we have that

$$\mathfrak{J}_{x_0}^{+}(g) = \rho_{x_0}^{\Gamma \oplus_x \mathcal{H}_A}; g = \mathfrak{m}_{\mathcal{H}_x}^{\Gamma \oplus_x \mathcal{H}_A}; g = \mathfrak{J}_{x_1}^{-}(g).$$

First suppose that $x \in t_l \Gamma$. Then $t_l \Gamma_m = x_1$. Thus,

$$t_l(\mathfrak{r}_x^{\Gamma})_{\Gamma_m} = (\mathfrak{r}_x^{\Gamma})_{t_l \Gamma_m} = (\mathfrak{r}_x^{\Gamma})_{x_1} = \mathrm{id}_A .$$

Hence, Lemma 3.4.2.33 tells us that $\mathfrak{J}_x^{+}(\mathfrak{r}_x^{\Gamma}; g) = (\mathfrak{r}_x^{\Gamma} \oplus_{x_1} \mathcal{H}_{\mathrm{id}_A}); \mathfrak{J}_{x_1}^{-}(g)$. Furthermore, it is easily verified that $(\mathfrak{r}_x^{\Gamma} \oplus_{x_1} \mathrm{id}_{\mathcal{H}_A}) = \mathfrak{r}_{x_0}^{\Gamma \oplus_x \mathcal{H}_A}$. Hence,

$$\begin{aligned}
\mathfrak{J}_x^{+}(\mathfrak{r}_x^{\Gamma}; g) &= (\mathfrak{r}_x^{\Gamma} \oplus_{x_1} \mathcal{H}_{\mathrm{id}_A}); \mathfrak{J}_{x_1}^{-}(g) \\
&= (\mathfrak{r}_{x_0}^{\Gamma \oplus_x \mathcal{H}_A}); \mathfrak{J}_{x_1}^{-}(g) \\
&= (\mathfrak{r}_{x_0}^{\Gamma \oplus_x \mathcal{H}_A}); \mathfrak{J}_{x_0}^{+}(g) \\
&= g.
\end{aligned}$$

Now suppose that $x \notin t_l \Gamma$. Suppose that $y : B \in \Gamma$, and that $t_l y = x$. Then,

$$\begin{aligned}
\rho_B; (\mathrm{id}_B \odot_l \mathfrak{r}_A) &= (\rho_B \odot_l \mathrm{id}_A); (\mathrm{id}_B \odot_l \mathfrak{r}_A) \\
&= \rho_B \odot_l \mathfrak{r}_A \\
&= ((\mathrm{id}_B \odot_l \mathcal{H}_A); \rho_B) \odot_l (\mathfrak{r}_A; \mathcal{H}_A) \\
&= (\mathrm{id}_B \odot_l \mathcal{H}_A \odot_l \mathfrak{r}_A); (\rho_B \odot_l \mathcal{H}_A)
\end{aligned}$$

However, we have that

$$\rho_x^\Gamma = \bigodot_{y:B\in\Gamma} \begin{cases} \rho_B & \text{if } t_l y = x \\ \mathrm{id}_B & \text{otherwise} \end{cases}$$

$$\mathfrak{r}_x^\Gamma = \bigodot_{y:B\in\Gamma} \begin{cases} \mathrm{id}_B \odot_l \mathfrak{r}_A & \text{if } t_l y = x \\ \mathrm{id}_B & \text{otherwise} \end{cases}$$

Furthermore, the following identities are easily verified:

$$\mathfrak{r}_{x_1}^{\Gamma\oplus_x\mathcal{H}_A} = \bigodot_{y:B\in\Gamma} \begin{cases} \mathrm{id}_B \odot_l \mathrm{id}_{\mathcal{H}_A} \odot_l \mathfrak{r}_A & \text{if } t_l y = x \\ \mathrm{id}_B & \text{otherwise} \end{cases}$$

$$\rho_{x_0}^{\Gamma\oplus_x\mathcal{H}_A} = \bigodot_{y:B\in\Gamma} \begin{cases} \rho_B \odot_l \mathcal{H}_A & \text{if } t_l y = x \\ \mathrm{id}_B & \text{otherwise} \end{cases}$$

Hence,

$$\begin{aligned} \mathfrak{J}_x^-(\mathfrak{r}_x^\Gamma; g) &= \rho_x^\Gamma; \mathfrak{r}_x^\Gamma; g \\ &= \mathfrak{r}_{x_1}^{\Gamma\oplus_x\mathcal{H}_A}; \rho_{x_0}^{\Gamma\oplus_x\mathcal{H}_A}; g \\ &= \mathfrak{r}_{x_1}^{\Gamma\oplus_x\mathcal{H}_A}; \mathfrak{J}_{x_0}^+(g) \\ &= \mathfrak{r}_{x_1}^{\Gamma\oplus_x\mathcal{H}_A}; \mathfrak{J}_{x_1}^-(g) \\ &= g \end{aligned}$$

as required. □

Hence, $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ is a globular multicategory with strict homomorphism types. This assignment extends straightforwardly to the arrows and 2-cells of GlobMult, and in this way we obtain a strict 2-functor:

$$\mathrm{Mod}^{\{l\}} : \mathrm{GlobMult} \longrightarrow \mathrm{GlobMult}_{\mathcal{H}}^{\{l\}}\,.$$

Theorem 3.4.2.39. *For each $l \geq 0$, the functor $\mathrm{Mod}^{\{l\}}$ is strictly right adjoint to the functor $U_{\mathcal{H}}^{\{l\}} : \mathrm{GlobMult}_{\mathcal{H}}^{\{l\}} \to \mathrm{GlobMult}$ that forgets strict homomorphism types at level l.*

$$\begin{tikzcd} \mathrm{GlobMult}_{\mathcal{H}}^{\{l\}} \arrow[r, bend left, "U_{\mathcal{H}}^{\{l\}}"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult} \arrow[l, bend left, "\mathrm{Mod}^{\{l\}}"] \end{tikzcd}$$

Proof. Suppose that $\mathbb{X}$ is a globular multicategory with strict homomorphism types. Suppose that M is an n-type in $\mathbb{X}$, and $f : \Gamma \to M$ is an n-term in $\mathbb{X}$. We define the unit $\eta_{\mathbb{X}} : \mathbb{X} \to \operatorname{Mod}^{\{l\}} U_{\mathcal{H}}^{\{l\}}\mathbb{X}$, for each n-type $\eta_{\mathbb{X}}(M)$, and each n-term $\eta_{\mathbb{X}}(f)$ in $\operatorname{Mod}^{\{l\}} U_{\mathcal{H}}^{\{l\}}\mathbb{X}$ so that $\eta_{\mathbb{X}}(M)_0 = M$, and $\eta_{\mathbb{X}}(f)_0 = f$. The remaining structure is defined by induction on n. When $n < l$, we define $\eta_{\mathbb{X}}(M)$ to be the identity map on types and terms.

Now suppose that $n = l$, and suppose that M is an l-type in $\mathbb{X}$. Suppose that $m : M$ is the unique l-variable in $[M]$.We define

$$\begin{aligned}\mathcal{H}_{\eta_{\mathbb{X}}(M)} &= \mathcal{H}_M,\\ \mathfrak{m}_{\eta_{\mathbb{X}}(m)} &= \mathfrak{J}_m(\mathrm{id}_M).\end{aligned}$$

and

$$\mathcal{H}_{\eta_{\mathbb{X}}(f)} = \mathfrak{J}_{\Gamma(l)}(f; \mathfrak{r}_M).$$

Now suppose that $n > l$ and that M is an n-type in $\mathbb{X}$. Suppose that $m : M$ is the unique m-variable in $[M]$. Then we define

$$\begin{aligned}\lambda_{\eta_{\mathbb{X}}(M)} &= \mathfrak{J}_{s_l m}(\mathrm{id}_{M_0})\\ \rho_{\eta_{\mathbb{X}}(M)} &= \mathfrak{J}_{t_l m}(\mathrm{id}_{M_0})\end{aligned}$$

The laws for homomorphism types at level l, imply that this data satisfy the required properties, and that $\eta_{\mathbb{X}}$ is a natural homomorphism that preserves homomorphism types at level l.

Suppose that $\mathbb{Y}$ is a globular multicategory. Suppose that N is an n-type, and that $g : \Delta \to N$ is an n-term in $U_{\mathcal{H}}^{\{l\}} \operatorname{Mod}^{\{l\}} \mathbb{Y}$. The counit $\epsilon_{\mathbb{Y}}$ is defined so that $\epsilon_{\mathbb{Y}}(N) = N_0$ and $\epsilon_{\mathbb{Y}}(g) = g_0$.

The first identity triangle identity says that, whenever $\mathbb{X}$ has homomorphism types at level l, the composite assignment

$$\begin{tikzcd} U_{\mathcal{H}}^{\{l\}}\mathbb{X} \arrow[r, "U_{\mathcal{H}}^{\{l\}}\eta_{\mathbb{X}}"] & U_{\mathcal{H}}^{\{l\}} \operatorname{Mod}^{\{l\}} U_{\mathcal{H}}^{\{l\}}\mathbb{X} \arrow[r, "\epsilon_{U_{\mathcal{H}}^{\{l\}}\mathbb{X}}"] & U_{\mathcal{H}}^{\{l\}}\mathbb{X} \\ M \arrow[r, mapsto] & (M, \mathcal{H}_M, \ldots) \arrow[r, mapsto] & M \end{tikzcd}$$

is the identity assignment. The second triangle identity says that, for each globular multicategory $\mathbb{Y}$, the composite assignment

$$\begin{tikzcd} \operatorname{Mod}^{\{l\}} \mathbb{Y} \arrow[r, "\eta_{\operatorname{Mod}^{\{l\}} \mathbb{Y}}"] & \operatorname{Mod}^{\{l\}} U_{\mathcal{H}}^{\{l\}} \operatorname{Mod}^{\{l\}} \mathbb{Y} \arrow[r, "\operatorname{Mod}^{\{l\}} \epsilon_{\mathbb{Y}}"] & \operatorname{Mod}^{\{l\}} \mathbb{Y} \\ (M, \mathcal{H}_M, \ldots) \arrow[r, mapsto] & ((M, \mathcal{H}_M, \ldots), (\mathcal{H}_M, \ldots), \ldots) \arrow[r, mapsto] & (M, \mathcal{H}_M, \ldots) \end{tikzcd}$$

is the identity assignment. □

Example 3.4.2.40. These results also hold for n-globular multicategories, for finite n. Restricting to the case where $\dim \mathbb{X} = 1$ and $l = 0$, we recover Theorem 3.4.0.3.

Proposition 3.4.2.41. *Suppose that $l' < l$. Then* $\mathrm{Mod}^{\{l\}}$ *preserves representability at level l'. Furthermore, if a globular multicategory $\mathbb{X}$ is representable up to level $(l+1)$, then* $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ *is representable up to level l.*

Proof. First suppose that $\mathbb{X}$ is representable at level l', for some $l' < l$. Suppose that Γ is an l'-context in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. Suppose that $f : \Gamma \to M$ is an n-term in $\mathbb{X}$. First suppose that $n = l$. Then, we can define $\mathcal{H}_f = f; \mathfrak{r}_M$, and so f can be seen as an n-term in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. On the other hand, when $n > l$, the equivariance laws are vacuously satisfied since Γ does not contain any l-types. Hence, in this case also, f is an n-term in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. It follows from this analysis that if $\mathbb{X}$ is representable at level l', then $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ is representable at level l'.

Now suppose that $\mathbb{X}$ is representable up to level $(l+1)$. Let Γ be an l-context in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. Then, we define $\bigotimes\Gamma$ by

$$(\bigotimes\Gamma)_0 = \bigotimes\Gamma_0, \qquad \mathcal{H}_{\bigotimes\Gamma} = \bigotimes\mathcal{H}_\Gamma,$$

and

$$\mathfrak{m}_{\bigotimes\Gamma} = \bigotimes\mathfrak{m}_\Gamma, \qquad \mathfrak{r}_{\bigotimes\Gamma} = \bigotimes\mathfrak{r}_\Gamma.$$

We define the compositor $\mathbf{m}_\Gamma : \Gamma \to \bigotimes\Gamma$ by

$$(\mathbf{m}_\Gamma)_0 = \mathbf{m}_{\Gamma_0}, \qquad \mathcal{H}_{\mathbf{m}_\Gamma} = \mathbf{m}_{\mathcal{H}_\Gamma}$$

We have that

$$\begin{aligned}(\mathbf{m}_\Gamma)_0; \mathfrak{r}_{\bigotimes\Gamma} &= \mathbf{m}_{\Gamma_0}; \bigotimes\mathfrak{r}_\Gamma\\ &= \mathfrak{r}_\Gamma; \mathbf{m}_{\mathcal{H}_\Gamma}\\ &= \mathfrak{r}_\Gamma; \mathcal{H}_{\mathbf{m}_\Gamma},\end{aligned}$$

and

$$\begin{aligned}(\mathbf{m}_\Gamma)_0; \mathfrak{m}_{\bigotimes\Gamma} &= \mathbf{m}_{\Gamma_0}; \bigotimes\mathfrak{m}_\Gamma\\ &= \mathfrak{m}_\Gamma; \mathbf{m}_{\mathcal{H}_\Gamma}\\ &= \mathfrak{m}_\Gamma; \mathcal{H}_{\mathbf{m}_\Gamma}.\end{aligned}$$

Hence, $\mathbf{m}_\Gamma$ is an l-term in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. It is a compositor of Γ by construction. □

Proposition 3.4.2.42. *Suppose that $l' < l$. Then the functor* $\mathrm{Mod}^{\{l\}} : \mathrm{GlobMult} \to \mathrm{GlobMult}$ *preserves coproducts at level l'.*

Proof. Suppose that $\mathbb{X}$ has coproducts at level l'. Let $\{A_i : sA \nrightarrow tA\}_{i \in I}$ be a set of parallel l-types in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. Since $l' < l$, we can define $\coprod_{i\in I} A_I : sA \nrightarrow tA$ by $(\coprod_{i\in I} A_I)_0 = \coprod_{i\in I}(A_i)_0$. Furthermore, each inclusion $\iota_i : (A_i)_0 \to \coprod_{i\in I}(A_i)_0$ defines an l'-term in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. Suppose that Γ is a π-shaped n-context in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ such that, for some l-variable x in Γ, we have that $\Gamma_x = \coprod_{i\in I} A_I$. Suppose that x is not the source or target of any other variable in Γ. Suppose that B is an n-type, and that $g : s\Gamma \to sB$, $h : t\Gamma \to tB$ are term-wise parallel $(n-1)$-terms. Suppose that for each $i \in I$, we have an n-term

$$f_i : \Gamma[A_i/x] \longrightarrow B, \quad s_{n-1}\iota_i^{\Gamma}; g \nrightarrow t_{n-1}\iota_i^{\Gamma}; h.$$

We will define a term $f_I : \Gamma \to B$ in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ such that $(f_I)_0$ is the term in $\mathbb{X}$ corresponding to the family $\{(f_i)_0\}_{i\in I}$. When $n < l$, it is clear that the term f_I exists and satisfies the required universal properties.

Now suppose that $n = l$. Since $l' < l$, we have that

$$(\mathcal{H}_\Gamma)[A_i/x] = \mathcal{H}_{\Gamma[A_i/x]}$$

Hence, we define $\mathcal{H}_{f_I} : \mathcal{H}_\Gamma \to \mathcal{H}_B,\ f_I \nrightarrow f_I$ to be the term in $\mathbb{X}$ corresponding to the family $\{\mathcal{H}_{f_i}\}_{i\in I}$. It follows that, for each $i \in I$, and for each variable $y : C \in \mathcal{H}_\Gamma$,

$$\begin{aligned}(\mathfrak{m}_{\Gamma[A_i/x]}; \iota_i^{\mathcal{H}_\Gamma})_y \ &= \begin{cases} \mathfrak{m}_B & \text{if } \dim y = l \\ \mathrm{id}_B & \text{if } \dim y \neq l \text{ and } y \neq x \\ \iota_i & \text{if } \dim y \neq l \text{ and } y = x \end{cases} \\ &= (\iota_i^{\mathcal{H}_\Gamma}; m_\Gamma)_y\end{aligned}$$

Hence, for each $i \in I$,

$$\begin{aligned}\iota_i^{\mathcal{H}_\Gamma}; \mathfrak{m}_\Gamma; \mathcal{H}_{f_I} \ &= \mathfrak{m}_{\Gamma[A_i/x]}; \iota_i^{\mathcal{H}_\Gamma}; \mathcal{H}_{f_I} \\ &= \mathfrak{m}_{\Gamma[A_i/x]}; \mathcal{H}_{f_i} \\ &= \mathcal{H}_{f_i}; \mathfrak{m}_B \\ &= \iota^{\mathcal{H}_\Gamma}; \mathcal{H}_f; \mathfrak{m}_B.\end{aligned}$$

Thus, $\mathfrak{m}_\Gamma; \mathcal{H}_{f_I} = \mathcal{H}_{f_I}; \mathfrak{m}_B$. Similarly, we have that $\mathfrak{r}_{\Gamma[A_i/x]}; \iota_i^{\mathcal{H}_\Gamma} = \iota_i^{\Gamma}; \mathfrak{r}_\Gamma$, and this implies that $\mathfrak{r}_\Gamma; \mathcal{H}_{f_I} = f_I; \mathfrak{r}_B$. The required universal property is now easily verified.

Now suppose that $n > l$. Since $l' < l$, for each l-variable $y \in \Gamma$, and each $i \in I$, we have that

$$\iota^i; \mathfrak{J}_y^+(f_I) = \mathfrak{J}_y^+(\iota^i; f_i) \ = \mathfrak{J}_y^-(\iota^i; f_i) = \iota^i; \mathfrak{J}_y^-(f_I)$$

Hence $\mathfrak{J}_y^+(f_I) = \mathfrak{J}_y^-(f_I)$, and so f_I is a term in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$. The required universal property follows immediately. □

3.4.3 Composing Level-wise Modules

We now show how we can compose modules construction at each level, in order to obtain right adjoints to $U^S_{\mathcal{H}} : \mathrm{GlobMult}^S_{\mathcal{H}} \to \mathrm{GlobMult}$, for more general sets $S \subseteq \omega$.

Proposition 3.4.3.1. *Suppose that $l < j$. Then $\mathrm{Mod}^{\{l\}}$ preserves homomorphism types at level j.*

Proof. Suppose that $\mathbb{X}$ is a globular multicategory with homomorphism types at level j. Let M be a j-type. Suppose that $m : M$ is the unique n-variable in $[M_0]$. First suppose that $n = j$. Then we define the $(j+1)$-type $\mathcal{H}_M$ in $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ by:

$$(\mathcal{H}_M)_0 = \mathcal{H}_{M_0}, \qquad \lambda^l_{\mathcal{H}_M} = \mathfrak{J}_m(\lambda^l_M; \mathfrak{r}_{M_0}), \qquad \rho^l_{\mathcal{H}_M} = \mathfrak{J}_m(\rho^l_M; \mathfrak{r}_{M_0}),$$

and we define $\mathfrak{r}_M$ by

$$(\mathfrak{r}_M)_0 = \mathfrak{r}_{M_0},$$

For any j-term f,we define

$$(\mathcal{H}_f)_0 = \mathcal{H}_{f_0}.$$

For any $n > j$, and n-term g, we define

$$\mathfrak{J}_x(g) = \mathfrak{J}_x(g_0).$$

The required identities can be verified by plugging in reflexivity terms at levels l and j, and applying $\mathfrak{J}$. For example, we have that

$$\begin{aligned}
\mathfrak{r}_M; (\mathrm{id}_M \odot_l \mathfrak{r}_{t_l M}); \lambda^l_{\mathcal{H}_M} &= (\mathfrak{r}_M \odot_l \mathrm{id}_{t_l M}); (\mathrm{id}_M \odot_l \mathfrak{r}_{t_l M}); \mathfrak{J}_m(\lambda^l_M; \mathfrak{r}_M) \\
&= (\mathfrak{r}_M \odot_l \mathfrak{r}_{t_l M}); \mathfrak{J}_m(\lambda^l_M; \mathfrak{r}_M) \\
&= (\mathrm{id}_M \odot_l \mathfrak{r}_{t_l M}); (\mathfrak{r}_M \odot_l \mathrm{id}_{\mathcal{H}_M}); \mathfrak{J}_m(\lambda^l_M; \mathfrak{r}_M) \\
&= (\mathrm{id}_M \odot_l \mathfrak{r}_{t_l M}); \lambda^l_M; \mathfrak{r}_M \\
&= \mathfrak{r}_M \\
&= \mathfrak{r}_M; \mathrm{id}_{\mathcal{H}_M},
\end{aligned}$$

and so $(\mathrm{id}_M \odot_l \mathfrak{r}_{t_l M}); \lambda^l_{\mathcal{H}_M} = \mathrm{id}_{\mathcal{H}_M}$. □

Corollary 3.4.3.2. *Suppose that $S \subseteq \omega$ is a finite set. Then, we have an adjunction*

$$\begin{tikzcd}
\mathrm{GlobMult}^S_{\mathcal{H}} \arrow[r, bend left, "U^S_{\mathcal{H}}"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult} \arrow[l, bend left, "\mathrm{Mod}^S"]
\end{tikzcd}$$

Proof. Suppose that $S = \{l_1 < l_2 < \ldots < l_m\}$. Let $\mathcal{G}_i = \mathrm{GlobMult}_{\mathcal{H}}{}^{\{l_i,\ldots,l_m\}}$. Then we define Mod^S to be the following composite:

$$\mathrm{GlobMult} = \mathcal{G}_n \xrightarrow{\mathrm{Mod}^{\{l_m\}}} \mathcal{G}_{n-1} \xrightarrow{\mathrm{Mod}^{\{l_{m-1}\}}} \cdots \xrightarrow{\mathrm{Mod}^{\{l_1\}}} \mathcal{G}_0 = \mathrm{GlobMult}^S_{\mathcal{H}}$$

□

Remark 3.4.3.3. It follows that a type (or term) in $\mathrm{Mod}^{[l]}\,\mathbb{X}$ has the data of a type (or term) in $\mathrm{Mod}^{\{i\}}$ for all or all $0 \leq i \leq l$.

In particular, when $\mathbf{d} = n$ is finite, and $S = [n]$, we obtain:

Corollary 3.4.3.4. *We have an adjunction*

$$\begin{tikzcd} n\text{ - }\mathrm{GlobMult}_{\mathcal{H}} \arrow[r, bend left, "U_{\mathcal{H}}"] \arrow[r, phantom, "\perp"] & n\text{ - }\mathrm{GlobMult} \arrow[l, bend left, "\mathrm{Mod}_n"] \end{tikzcd}$$

Proposition 3.4.3.5. *Let $S \subseteq [n]$ be a finite collection of levels. Then the forgetful functor*

$$U^S_{\mathcal{H}} : \mathrm{GlobMult}^S_{\mathcal{H}} \longrightarrow \mathrm{GlobMult}$$

is monadic and comonadic.

Proof. Since $U^S_{\mathcal{H}}$ forgets essentially algebraic data, it is clearly conservative. The claim now follows from the (Cat-enriched) (co)monadicity theorem since $U^S_{\mathcal{H}}$ has a left and a right adjoint and since $\mathrm{GlobMult}^S_{\mathcal{H}}$ and $\mathrm{GlobMult}$ are locally presentable. □

The following proposition allows us to prove the case when S is infinite.

Proposition 3.4.3.6. *Suppose that l is finite. Then we have an adjunction:*

$$\begin{tikzcd} \mathrm{GlobMult}^{[l]}_{\mathcal{H}} \arrow[r, bend left, "U^{\{l\}}_{\mathcal{H}}"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult}^{[l-1]}_{\mathcal{H}} \arrow[l, bend left, "\mathrm{Mod}^{\{l\}}"] \end{tikzcd}$$

Proof. We have the following commutative triangle:

$$\begin{tikzcd} \mathrm{GlobMult}^{[l]}_{\mathcal{H}} \arrow[rr, "U^{\{l\}}_{\mathcal{H}}"] \arrow[dr, "U^{[l]}_{\mathcal{H}}"'] & & \mathrm{GlobMult}^{[l-1]}_{\mathcal{H}} \arrow[dl, "U^{[l-1]}_{\mathcal{H}}"] \\ & \mathrm{GlobMult} & \end{tikzcd}$$

By Proposition 3.4.3.5, the functor $U_{\mathcal{H}}^{[l-1]}$ is comonadic, and by Corollary 3.4.3.2, the functor $U_{\mathcal{H}}^{[l]}$ has a right adjoint. Furthermore, $\mathrm{GlobMult}_{\mathcal{H}}^{[l]}$ is locally finitely presentable since it is the category of models of an essentially algebraic theory. Hence, the (Cat-enriched) adjoint triangle theorem (see [18,33]) applies, and so $U_{\mathcal{H}}^{\{l\}}$ has a right adjoint as required.

Explicitly given a globular multicategory $\mathbb{X}$ with homomorphism types at level j for $j \leq l-1$, the right adjoint $\mathrm{Mod}^{\{l\}} : \mathrm{GlobMult}^{[l-1]} \to \mathrm{GlobMult}^{[l]}$ is defined so that $\mathrm{Mod}^{\{l\}}\,\mathbb{X}$ is the subobject of $\mathrm{Mod}^{[l]}\,U_{\mathcal{H}}^{[l]}\mathbb{X}$ whose homomorphism type data at level j agrees with that of $\mathbb{X}$ for each $j \leq l-1$. □

Corollary 3.4.3.7. *Suppose that $S \subseteq \omega$ is an infinite set. Then we have an adjunction*

$$\begin{tikzcd} \mathrm{GlobMult}_{\mathcal{H}}^{S} \arrow[r, bend left, "U_{\mathcal{H}}^{S}"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult} \arrow[l, bend left, "\mathrm{Mod}^{S}"] \end{tikzcd}$$

Proof. Suppose that $S = \{l_0 < \ldots < l_i < \ldots\}$. Let $S_i = \{l_0, \ldots, l_i\}$. First note that $\mathrm{GlobMult}_{\mathcal{H}}^{S}$ is the strict 2-limit of the following chain of forgetful functors,

$$\cdots \longrightarrow \mathrm{GlobMult}_{\mathcal{H}}{}^{S_1} \xrightarrow{U_{\mathcal{H}}^{S_1}} \mathrm{GlobMult}_{\mathcal{H}}{}^{S_0} \xrightarrow{U_{\mathcal{H}}^{S_0}} \mathrm{GlobMult}.$$

and that $U_{\mathcal{H}}^{S}$ is the coprojection from this limit to GlobMult. By Proposition 3.4.3.6, each of these forgetful functors is cocontinuous. It now follows that $U_{\mathcal{H}}^{S}$ is cocontinuous since a cone in $\mathrm{GlobMult}_{\mathcal{H}}$ amounts to a sequence of cones in $(\mathrm{GlobMult}_{\mathcal{H}}^{S_i})_{i\in\mathbb{N}}$. Since $\mathrm{GlobMult}_{\mathcal{H}}^{S}$ and GlobMult are locally presentable, the adjoint functor theorem implies that $U_{\mathcal{H}}^{S}$ has a right adjoint Mod^{S}. □

In particular, when $S = \omega$, we obtain:

Theorem 3.4.3.8. *The forgetful functor $U_{\mathcal{H}} : \mathrm{GlobMult}_{\mathcal{H}} \to \mathrm{GlobMult}$ has a right adjoint.*

$$\begin{tikzcd} \mathrm{GlobMult}_{\mathcal{H}} \arrow[r, bend left, "U_{\mathcal{H}}"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult} \arrow[l, bend left, "\mathrm{Mod}"] \end{tikzcd}$$

In order to describe the modules construction more explicitly, let $L^i : \mathrm{GlobMult} \to \mathrm{GlobMult}$ be the monad $U_{\mathcal{H}}^{[i]} L_{\mathcal{H}}^{[i]}$, and let $\mathcal{M}^i : \mathrm{GlobMult} \to \mathrm{GlobMult}$ be the comonad $U_{\mathcal{H}}^{[i]}\,\mathrm{Mod}^{[i]}$. Let $\beta^{\{i+1\}}$ be the unit of the adjunction $U_{\mathcal{H}}^{\{i+1\}} \vdash L_{\mathcal{H}}^{\{i+1\}}$. We have that $L^{i+1} = U^{[i]}_{\mathcal{H}} U^{\{i+1\}}_{\mathcal{H}} L^{\{i+1\}}_{\mathcal{H}} L^{[i]}_{\mathcal{H}}$. Hence, we define a natural transformation $\eta^i : L^i \Rightarrow L^{i+1}$ by

$$L^i \xrightarrow{U^{[i]}_{\mathcal{H}} \beta^{\{i+1\}} L^{[i]}_{\mathcal{H}}} L^{i+1}.$$

Let $y^{\{i+1\}}$ be the counit of the adjunction $\mathrm{Mod}^{\{i+1\}} \vdash U^{\{i+1\}}_{\mathcal{H}}$. Similarly, we have that $\mathcal{M}^{i+1} = U^{[i]}_{\mathcal{H}} U^{\{i+1\}}_{\mathcal{H}} \mathrm{Mod}^{\{i+1\}} \mathrm{Mod}^{[i]}$. Hence, we define a natural transformation $\epsilon^i : \mathcal{M}^{i+1} \Rightarrow \mathcal{M}^i$ by

$$\mathcal{M}^{i+1} \xrightarrow{U^{[i]}_{\mathcal{H}} y^{\{i+1\}} L^{[i]}_{\mathcal{H}}} \mathcal{M}^i.$$

It follows that η^i is a morphism of monads, and ϵ^i is a morphism of comonads. Consider the following diagram of monads:

$$\begin{tikzcd} L^{-1} \arrow[r, "\eta^{-1}"] & L^0 \arrow[r, "\eta^0"] & L^1 \arrow[r, "\eta^1"] & \cdots \end{tikzcd}$$

An algebra ϕ of the algebraic limit, L^ω, of this diagram consists of a globular multicategory with an L^i-algebra structure, ϕ_i, for each i respecting η^i. However, an algebra of L^i is a globular multicategory with homomorphism types at level l for each $l \leq i$. Furthermore, the requirement that $\eta^i \circ \phi^{i+1} = \phi^i$ says that, for each $l \leq i$, the homomorphism types at level l chosen by ϕ^{i+1} agree with those chosen by i. Hence, an algebra for L^ω is simply a globular multicategory with homomorphism types, and the forgetful functor of L^ω is $U_{\mathcal{H}} : \mathrm{GlobMult}_{\mathcal{H}} \to \mathrm{GlobMult}$. However, for any globular multicategory $\mathbb{X}$, the free algebra $L^\omega \mathbb{X}$ is just the colimit $\mathrm{colim}_i\, L^i \mathbb{X}$. Hence, by adjointness, we have a natural isomorphism

$$\begin{aligned} \mathrm{GlobMult}(\mathbb{X}, U_{\mathcal{H}} \mathrm{Mod}\, \mathbb{Y}) &\cong \mathrm{GlobMult}(L^\omega \mathbb{X}, \mathbb{Y}) \\ &= \mathrm{GlobMult}(\operatorname*{colim}_i L^i \mathbb{X}, \mathbb{Y}) \\ &\cong \mathrm{GlobMult}(\mathbb{X}, \lim_i \mathcal{M}^i \mathbb{Y}) \\ &= \mathrm{GlobMult}(\mathbb{X}, \mathcal{M}^\omega \mathbb{Y}) \end{aligned}$$

where $\mathcal{M}^\omega \mathbb{Y} = \operatorname*{colim}_i \mathcal{M}^i \mathbb{Y}$ is the colimit of the following diagram:

$$\begin{tikzcd} \mathcal{M}^{-1}\mathbb{Y} & \mathcal{M}^0\mathbb{Y} \arrow[l, "\epsilon^{-1}"'] & \mathcal{M}^1\mathbb{Y} \arrow[l, "\epsilon^0"'] & \cdots \arrow[l, "\epsilon^1"'] \end{tikzcd}$$

Since this isomorphism is natural in $\mathbb{X}$ and $\mathbb{Y}$, we have that $\mathcal{M}^\omega \cong U_{\mathcal{H}} \mathrm{Mod}$.

We refer to an n-type in $\mathrm{Mod}\, \mathbb{X}$ as an *n-module*, and an n-term in $\mathrm{Mod}\, \mathbb{X}$ as a *homomorphism* of n-modules. It follows from the description of $\mathrm{Mod}\, \mathbb{X}$ as a limit, that an n-module (or homomorphism) has the data of a type (or term) in $\mathrm{Mod}^{\{i\}}$ for all or all $0 \leq i \leq l$; that is:

- Given parallel $(n-1)$-modules sM, tM, an n-module $M : sM \nrightarrow tM$ in $\mathbb{X}$ consists of:
  - An n-type $M_0 : (sM)_0 \nrightarrow (tM)_0$
  - For each $j < n$, we require n-homomorphisms

$$\begin{aligned} \lambda^j_M &: \mathcal{H}_{t_n M} \odot_j M \longrightarrow M, & \lambda^j_{sM} &\nrightarrow \lambda^j_{tM}, \\ \rho^j_M &: M \odot_j \mathcal{H}_{t_j M} \longrightarrow M, & \rho^j_{sM} &\nrightarrow \rho^j_{tM} \end{aligned}$$

    satisfying laws making them compatible with the multiplication of M and compatible with each other.
  - an $(n+1)$-module $\mathcal{H}_M : M \nrightarrow M$
  - an $(n+1)$-homomorphism $\mathfrak{r}_M : M \to \mathcal{H}_M$, $\mathfrak{r}_{sM} \nrightarrow \mathfrak{r}_{tM}$
  - an $(n+1)$-homomorphism $m_M : M \odot_n M \nrightarrow M$ satisfying associativity and unit laws

  Here we take $\lambda^j_{sM} = \rho^j_{sM} = \mathrm{id}_{sM}$ and $\lambda^j_{tM} = \rho^j_{tM} = \mathrm{id}_{tM}$ when $j = n-1$ and we make similar definitions for $\mathfrak{r}$ and $\mathfrak{m}$.

- Given an n-context of modules Γ, an n-module M and parallel $(n-1)$-homomorphisms $sf : s\Gamma \to sM$ and $tf : t\Gamma \to tM$, an n-homomorphism $f : \Gamma \to M$, $sf \nrightarrow tf$ consists of:
  - An n-term $f_0 : \Gamma \to M$, $(sf)_0 \nrightarrow (tf)_0$ satisfying equivariance laws for all $j < n$
  - An $(n+1)$-homomorphism $\mathcal{H}_f : \Gamma \to M$, $f \nrightarrow f$ which respects the unit and multiplication homomorphisms of Γ and M

Our decomposition of this construction into level-wise parts shows that this definition is not circular.

Remark 3.4.3.9. Suppose that $l \geq 0$. Suppose that S is a set of levels such that $\operatorname{Min} S > l$. Then Proposition 3.4.2.41 implies that Mod^S preserves representability at level l. Furthermore, if a globular multicategory $\mathbb{X}$ is representable up to level $(\operatorname{Min} S + 1)$, then $\mathrm{Mod}^S\,\mathbb{X}$ is representable up to level $\operatorname{Min} S + 1$.

3.5 Enrichment

3.5.1 Level-wise Enrichment

We can describe a notion of level-wise enrichment by combining the level-wise modules construction with the level-wise families construction. Recall that for any globular multicategory $\mathbb{X}$, we define the families construction at level l as follows:

- When $n < l$, an n-type in $\mathrm{Fam}^{\{l\}}\,\mathbb{X}$ is an n-type in $\mathbb{X}$.
- An l-type in $\mathrm{Fam}^{\{l\}}\,\mathbb{X}$ is a set of l-types $\{M(\iota) \mid \iota \in \mathbf{I}_M\}$ in $\mathbb{X}$.
- When $n > l$, an n-type M consists of an n-type $M(\iota, \iota')$ for each $\iota \in I_{s_n M}$ and $\iota' \in I_{t_n M}$ such that when $n = l + 1$, we have that $M(\iota, \iota') : s_n M(\iota) \;\;\nrightarrow t_n M(\iota')$ and when $n > l + 1$, we have that $M(\iota, \iota') : sM(\iota, \iota') \;\;\nrightarrow tM(\iota, \iota')$.
- Similarly, when $n < l$, an n-term in $\mathrm{Fam}^{\{l\}}\,\mathbb{X}$ is an n-term in $\mathbb{X}$ and when $n \geq l$, an n-term in $\mathrm{Fam}^{\{l\}}\,\mathbb{X}$ is a family of n-terms of $\mathbb{X}$ indexed by a set which depends only on n-dimensional data.

Definition 3.5.1.1. Let $\mathrm{GlobMult}_{\mathcal{H}}^{[>l]}$ be the category of globular multicategories with homomorphism types at level j for each $j > l$. Similarly, let $\mathrm{GlobMult}_{\mathcal{H}}^{[\geq l]}$ be the category of globular multicategories with homomorphism types at level j for each $j \geq l$. Then for each l. we will define a functor $\mathbf{E}_l : \mathrm{GlobMult}^{[>l]} \to \mathrm{GlobMult}^{[\geq l]}$ which *enriches at level* l. As in the 1-dimensional case, we define:

$$\mathbf{E}_l = \mathrm{Mod}^{\{l\}} \circ \mathrm{Fam}^{\{l\}}$$

Unwrapping this definition, we find that an n-type X of $\mathbf{E}_n\mathbb{V}$ consists of:

- A set X_0 of n-types in $\mathbb{X}$
- For each pair of objects $A, B \in X_0$, an $(n+1)$-type $[A, B] : A \;\;\nrightarrow B$ in $\mathbb{V}$
- Composition and unit terms, $[A, B] \odot [B, C] \to [A, C]$ and $[A] \to [A, A]$, satisfying associativity and unit laws

Remark 3.5.1.2. Suppose that $i < n$. Then we have that $\mathrm{tr}_n \mathbf{E}_i \mathbb{X} = \mathbf{E}_i \mathrm{tr}_n \mathbb{X}$.

Remark 3.5.1.3. Whenever $l' < l$, the enrichment functor $\mathbf{E}_l$ preserves representability at level l'. This follows from Proposition 2.9.2.7 and Proposition 3.4.2.41.

Example 3.5.1.4. When $\mathbb{X}$ is a virtual double category, a 0-type in $\mathbf{E}_0\mathbb{X}$ is a category enriched in $\mathbb{X}$, as described by Leinster [31]. Similarly, a 0-term in $\mathbf{E}_0\mathbb{X}$ is a functor between such categories, as described ibid.

Example 3.5.1.5. Suppose that $\mathcal{C}$ is a strict monoidal category, seen as a one-object 2-category. Then $\mathbf{E}_0\,\mathrm{Sq}\,\mathcal{C}$ is the virtual double category of categories, and profunctors enriched in $\mathcal{C}$, described by [17].

Example 3.5.1.6. Suppose that $k \geq 0$. We say that a globular multicategory $\mathbb{X}$ is *k-terminal* when, for all $i < k$, the category $\mathrm{Type}\,\mathbb{X}(i)$ is the terminal category. Now suppose that $k > 0$, and that $\mathbb{X}$ is k-terminal. Then $\mathbf{E}_{k-1}\mathbb{X}$ is $(k-1)$-terminal. Every k-terminal globular multicategory is representable up to level $k-1$. Hence, $\mathbf{E}_{k-1}\mathbb{X}$ is representable up to level $k-2$. If, moreover, $\mathbb{X}$ is representable up to level k, then Proposition 2.9.2.7 and Proposition 3.4.2.41 imply that $\mathbf{E}_{k-1}\mathbb{X}$ is representable up to level $k-1$.

We say that an n-category $\mathcal{C}$ is *k-terminal*, when $\mathcal{C}$ has a unique i-cell for all $i < k$. It is well known that higher categories of this sort correspond to $(n-k)$-categories with a coherent choice of k monoidal structures. In this case, $\mathrm{Sq}\,\mathcal{C}$ is a representable k-terminal $(n-1)$-globular multicategory, and so $\mathbf{E}_{k-1}\,\mathrm{Sq}\,\mathcal{C}$ is a $(k-1)$-terminal n-globular multicategory representable up to level $(k-2)$.

Definition 3.5.1.7. Let $\mathrm{GlobMult}_{\top_k}$ be the category of k-terminal globular multicategories. There is a canonical functor

$$\mathrm{GlobMult}_{\top_{k+1}} \xrightarrow{\text{shift}} \mathrm{GlobMult}_{\top_k}$$

such that $\mathrm{shift}\,\mathbb{X} = \mathbb{X}(\star,\star)$ where $\star$ is the unique 0-type of $\mathbb{X}$; that is $\mathrm{shift}\,\mathbb{X}$ forgets the 0-types and terms of $\mathbb{X}$. An *ω-terminal* globular multicategory is an object in the limit of the following diagram:

$$\cdots \xrightarrow{\text{shift}} \mathrm{GlobMult}_{\top_k} \xrightarrow{\text{shift}} \cdots \mathrm{GlobMult}_{\top_0} = \mathrm{GlobMult}$$

This amounts to a choice of k-terminal globular multicategory $\mathbb{X}_k$, for each $k \geq 0$, such that $\mathrm{shift}\,\mathbb{X}_{k+1} = \mathbb{X}_k$. We denote the 2-category of ω-terminal globular multicategories by $\mathrm{GlobMult}_{\top_\omega}$.

A *symmetric monoidal globular multicategory* is an ω-terminal globular multicategory such that $\mathbb{X}_k$ is representable up to level k, for each k. We denote the 2-category of symmetric monoidal globular multicategories with chosen compositors and compositor preserving homomorphisms by $\mathrm{GlobMult}_{\mathrm{Sym}}$.

Example 3.5.1.8. Suppose that $\mathcal{C}$ is a symmetric monoidal category. Then, for each finite k, there is a k-terminal $k+1$-category $\mathbf{B}^k\mathcal{C}$ such that a k-cell of $\mathbf{B}^k\mathcal{C}$ is a 0-cell of $\mathcal{C}$, and an $(k+1)$-cell of $\mathbf{B}\mathcal{C}$ is an 1-cell of $\mathcal{C}$. Hence, we have a symmetric monoidal globular multicategory $\mathbb{V}_{\otimes}\mathcal{C}$ such that

$$(\mathbb{V}_{\otimes}\mathcal{C})_k = \operatorname{Sq}\mathbf{B}^k\mathcal{C}.$$

Let SymMonCat be the category of symmetric monoidal categories, and strict monoidal functors. We have defined the objects-part of a strict 2-functor,

$$\mathbb{V}_{\otimes} : \mathrm{SymMonCat} \longrightarrow \mathrm{GlobMult}_{\mathsf{Sym}}$$

Conversely, suppose that $\mathbb{X}$ is a symmetric monoidal globular multicategory. Let $\mathcal{V}_{\otimes}\mathbb{X}$ be the category $\operatorname{Type}\mathbb{X}_0(0) = \operatorname{Type}\mathbb{X}_1(1) = \operatorname{Type}\mathbb{X}_2(2) = \cdots$. Let $A, B \in \mathcal{V}_{\otimes}\mathbb{X}$. Let $\mathbf{m}_{A\odot_0 B} : A \odot_0 B \to A \otimes_0 B$ be a compositor in $\mathbb{X}_1$. Then we define the product $A \otimes B$ to be $A \otimes_0 B$. This defines a product functor

$$- \otimes - : \mathcal{V}_{\otimes}\mathbb{X} \times \mathcal{V}_{\otimes}\mathbb{X} \longrightarrow \mathcal{V}_{\otimes}\mathbb{X}$$

Since $A \otimes_0 B$ can be seen as the target term of a compositor $\mathcal{A} \odot_k \mathcal{B} \to A \otimes_k B$ in $\mathbb{X}_{k+1}$ for each $k \geq 0$, an Eckmann-Hilton argument implies that this product is symmetric. Hence, we have defined the objects-part of a strict 2-functor

$$\mathcal{V}_{\otimes} :\ \mathrm{GlobMult}_{\mathsf{Sym}} \longrightarrow \mathrm{SymMonCat}\,.$$

We have a strict 2-adjunction:

$$\begin{tikzcd}
\mathrm{SymMonCat} \arrow[r, bend left, "\mathbb{V}_{\otimes}"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult}_{\mathsf{Sym}} \arrow[l, bend left, "\mathcal{V}_{\otimes}"]
\end{tikzcd}$$

whose unit is an equivalence.

Example 3.5.1.9. Suppose that $\mathbb{X}$ is an ω-terminal globular multicategory. Then, following Example 3.5.1.6, we can define an ω-terminal globular multicategory $\mathbf{E}\mathbb{X}$ by

$$(\mathbf{E}\mathbb{X})_k = \mathbf{E}_k\mathbb{X}_{k+1}$$

This assignment defines a functor

$$\mathrm{GlobMult}_{\top_\omega} \xrightarrow{\mathbf{E}} \mathrm{GlobMult}_{\top_\omega}\,.$$

If $\mathbb{X}$ is symmetric monoidal, then $\mathbf{E}\mathbb{X}$ is also symmetric monoidal. Let $\mathbf{E} : \mathrm{SymMonCat} \to \mathrm{SymMonCat}$ be the enrichment functor on symmetric monoidal categories. Then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{SymMonCat} \arrow[r, "\mathbf{E}"] \arrow[d, "\mathbb{V}"] & \mathrm{SymMonCat} \\
\mathrm{GlobMult}_{\mathrm{Sym}} \arrow[r, "\mathbf{E}"] \arrow[d, "\mathcal{V}"] & \mathrm{GlobMult}_{\mathrm{Sym}} \arrow[u, "\mathcal{V}"'] \arrow[d, "\mathcal{V}"] \\
\mathrm{SymMonCat} \arrow[r, "\mathbf{E}"] & \mathrm{SymMonCat}
\end{tikzcd}$$

Hence, our notion of enrichment agrees with the usual notion of enrichment of symmetric monoidal categories.

3.5.2 Iterated Strict Enrichment

The modules construction is closely connected to the process of iterated enrichment. In this section, we describe this correspondence, and use it to formalize many of the previously-mentioned intuitions about the n-modules construction. It is worth pointing out that in order to iteratively enrich, we must first shift the level of enrichment: the collection of categories enriched in "categories enriched in $\mathbb{X}$ at level $k+1$" with its monoidal (or multicategory structure) is described by $\mathbf{E}_k\mathbf{E}_{k+1}\mathbb{X}$. More generally, whenever $k + l < n$, we define *l-fold enrichment at level k* to be the composite:

$$\mathbf{E}_k\mathbf{E}_{k+1}\cdots\mathbf{E}_{k+l}$$

Example 3.5.2.1. Suppose that $\mathbb{X}$ is an ω-terminal globular multicategory. Then,

$$(\mathbf{E}\mathbf{E}\mathbb{X})_k = \mathbf{E}_k\mathbf{E}_{k+1}\mathbb{X}_{k+2}$$

Thus, the need to change level is absent in this case, since it is implicitly handled by the shift functor. Applying the results of Example 3.5.1.9, we have that

$$\mathcal{V}_{\otimes}(\mathbf{E}^n\mathbb{X}) \cong \mathbf{E}^n\mathcal{V}_{\otimes}\mathbb{X}, \qquad \mathbf{E}^n\mathcal{C} \cong \mathcal{V}_{\otimes}\mathbf{E}^n\mathbb{V}_{\otimes}\mathcal{C}$$

Hence, our notion of iterated enrichment agrees with the usual notion of iterated enrichment of symmetric monoidal categories.

Inspecting the description of the families construction above, the following result becomes apparent:

Proposition 3.5.2.2. *Suppose that $l < j$. Then, we have that* $\mathrm{Fam}^{\{l\}} : \mathrm{GlobMult} \to \mathrm{GlobMult}$ *preserves homomorphism types at level j.*

This allows us to describe situations in which families constructions and modules constructions commute:

Proposition 3.5.2.3. *Suppose that $i < j$. Then the following square commutes up to natural isomorphism:*

$$\begin{tikzcd}
\mathrm{GlobMult} \arrow[r, "\mathrm{Mod}^{\{j\}}"] \arrow[d, "\mathrm{Fam}^{\{i\}}"'] & \mathrm{GlobMult}^{\{j\}}_{\mathcal{H}} \arrow[d, "\mathrm{Fam}^{\{i\}}"] \arrow[dl, phantom, "\cong\Uparrow \phi_{i,j}"] \\
\mathrm{GlobMult} \arrow[r, "\mathrm{Mod}^{\{j\}}"'] & \mathrm{GlobMult}^{\{j\}}_{\mathcal{H}}
\end{tikzcd}$$

Remark 3.5.2.4. Note that in order for this result to hold up to isomorphism, and not just equivalence, we need to use a definition of $\mathrm{Fam}^{\{i\}}$ which uses a particular choice of one-element indexing families, and pullbacks, as opposed to allowing isomorphic indexing families.

Proof. Let $n < \omega$. First suppose that $n < i$. Then $\mathrm{Mod}^{\{j\}}\,\mathbb{Y}(n) = \mathbb{Y}(n) \cong \mathrm{Fam}^{\{i\}}\,\mathbb{Y}(n)$ for any globular multicategory $\mathbb{Y}$, and so we have an isomorphism

$$\mathrm{Mod}^{\{j\}}\,\mathrm{Fam}^{\{i\}}\,\mathbb{X}(n) \cong \mathbb{X}(n) \cong \mathrm{Fam}^{\{i\}}\,\mathrm{Mod}^{\{j\}}\,\mathbb{X}(n).$$

Now suppose that $i \leq n < j$. Then, for any globular multicategory $\mathbb{Y}$, $\mathrm{Fam}^{\{i\}}\,\mathbb{Y}(n)$ does not depend on any levels of $\mathbb{Y}$ above n; that is $\mathrm{Fam}^{\{i\}}\,\mathbb{Y}(n) = (\mathrm{Fam}^{\{i\}}\,\mathrm{tr}_n)\mathbb{Y}(n)$. Furthermore, $\mathrm{tr}_n\,\mathrm{Mod}^{\{j\}}\,\mathbb{X} \cong \mathbb{X}$. Hence,

$$\begin{aligned}
\mathrm{Mod}^{\{j\}}\,\mathrm{Fam}^{\{i\}}\,\mathbb{X}(n) &= \mathrm{Fam}^{\{i\}}\,\mathbb{X}(n) \\
&= \mathrm{Fam}^{\{i\}}\,\mathrm{tr}_n\mathbb{X}(n) \\
&\cong \mathrm{Fam}^{\{i\}}\,\mathrm{tr}_n\,\mathrm{Mod}^{\{j\}}\,\mathbb{X}(n) \\
&= \mathrm{Fam}^{\{i\}}\,\mathrm{Mod}^{\{j\}}\,\mathbb{X}(n)
\end{aligned}$$

Finally, suppose that $j \leq n$. Then an n-type M of $\mathrm{Mod}^{\{j\}}\,\mathrm{Fam}^{\{i\}}\,\mathbb{X}$ consists of a pair of sets of i-types $\{s_iM(\iota) \mid \iota \in \mathbf{I}_{s_iM}\}$, and $\{t_iM(\iota) \mid \iota \in \mathbf{I}_{t_iM}\}$ in $\mathbb{X}$ together with, for each $\iota \in \mathbf{I}_{s_iM}, \iota' \in \mathbf{I}_{t_iM}$:

- An n-type $M(\iota, \iota')$ in $\mathbb{X}$ such that $s_iM(\iota, \iota') = (s_iM)(\iota)$, and $t_iM(\iota, \iota') = (t_iM)(\iota')$,
- Data making $M(\iota, \iota')$ an n-type in $\mathrm{Mod}^{\{j\}}\,\mathbb{X}$

On the other hand, an n-type in $\mathrm{Fam}^{\{i\}}\,\mathrm{Mod}^{\{j\}}\,\mathbb{X}$ consists of a pair of sets of i-types $\{s_iM(\iota) \mid \iota \in \mathbf{I}_{s_iM}\}$ and $\{t_iM(\iota) \mid \iota \in \mathbf{I}_{t_iM}\}$ in $\mathrm{Mod}^{\{j\}}\,\mathbb{X}(i) = \mathbb{X}(i)$ together with, for each $\iota \in I_{s_iM}, \iota' \in I_{t_iM}$, an n-type $M(\iota,\iota')$ in $\mathrm{Mod}^{\{j\}}\,\mathbb{X}$. This is exactly the same data described by an n-type in $\mathrm{Mod}^{\{j\}}\,\mathrm{Fam}^{\{i\}}\,\mathbb{X}$. The terms of $\mathrm{Mod}^{\{j\}}$ and $\mathrm{Fam}^{\{i\}}$ can be compared in a similar manner. These comparisons induce the required natural isomorphism. □

This result implies the following corollary, which allows us to see iterated enrichment as the composite of a families construction with a modules construction:

Corollary 3.5.2.5. *We have a natural isomorphism,*

$$\mathbf{E}_k\mathbf{E}_{k+1}\cdots\mathbf{E}_{k+l} \cong \mathrm{Mod}^{\{k,\ldots,k+l\}}\,\mathrm{Fam}^{\{k,\ldots,k+l\}}$$

Example 3.5.2.6. These results also work for finite n. In this case, we have that

$$\mathrm{Mod}\,\mathrm{Span}(\mathrm{Set}_n) \cong \mathrm{Mod}^{[n]}\,\mathrm{Fam}^{[n]}\,\mathbb{1}_n \cong \mathbf{E}_0\mathbf{E}_1\cdots\mathbf{E}_n\mathbb{1}.$$

Hence, the collection of higher modules in Set can be seen as the result of iteratively enriching starting with the terminal object. Since $\mathbb{1}_n = \mathbb{V}_{\otimes}\top$ is symmetric monoidal, we have that

$$\mathcal{V}_{\otimes}\mathbf{E}^n\mathbb{1} = \mathcal{V}_{\otimes}\mathbf{E}^n\top$$

Thus, the symmetric monoidal category of 0-types in $\mathrm{Mod}\,\mathrm{Span}(\mathrm{Set}_n)$ is the symmetric monoidal category of n-categories.

3.5.3 Infinitely Iterated Enrichment

We now consider the infinitely-iterated enrichment. Let $U_{\amalg} : \mathrm{GlobMult}_{\amalg} \to \mathrm{GlobMult}$ be the functor forgetting coproducts at all levels. Suppose that $n \geq -1$. Let $\mathbf{E}^nU_{\amalg} = \mathbf{E}_0\cdots\mathbf{E}_n : \mathrm{GlobMult}_{\amalg} \to \mathrm{GlobMult}$. Then the counits of the adjunctions defining Fam and Mod induce a canonical natural transformation:

$$\epsilon^{\mathbf{E}}_n : \mathbf{E}^{n+1} \implies \mathbf{E}^n$$

In order to make this precise, we define the natural transformation $y^{\mathbf{E}}_n : \mathbf{E}_{n+1}U_{\amalg} \Rightarrow$

$U_{\amalg}$ to be the following composite:

$$\begin{tikzcd}
\mathbf{E}_{n+1}U_{\amalg} \arrow[d, equals] \\
\mathrm{Mod}^{\{n+1\}}\, U_{\amalg}^{\{n+1\}}\, \mathrm{Fam}^{\{n+1\}}\, U_{\amalg}^{\{n+1\}} U_{\amalg}^{\{0,\ldots,n,n+2,\ldots\}} \arrow[d, Rightarrow, "\mathrm{Mod}^{\{n+1\}}\, U_{\amalg}^{\{n+1\}} y_{\amalg}^{n+1} U_{\amalg}^{\{0,\ldots,n,n+2,\ldots\}}"] \\
\mathrm{Mod}^{\{n+1\}}\, U_{\amalg}^{\{n+1\}} U_{\amalg}^{\{0,\ldots,n,n+2,\ldots\}} \arrow[d, equals] \\
\mathrm{Mod}^{\{n+1\}}\, U_{\amalg} \arrow[d, Rightarrow, "y_{\mathcal{H}}^{n+1} U_{\amalg}"] \\
U_{\amalg}
\end{tikzcd}$$

Then we define $\epsilon_n^{\mathbf{E}} : \mathbf{E}_0 \cdots \mathbf{E}_{n+1} U_{\amalg} \Rightarrow \mathbf{E}_0 \cdots \mathbf{E}_n U_{\amalg}$ to be $\mathbf{E}_0 \cdots \mathbf{E}_n y_n^{\mathbf{E}}$. Hence, whenever $\mathbb{X}$ is a globular multicategory with colimits, we can consider the limit of the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\epsilon_2^{\mathbf{E}}"] & \mathbf{E}^2\mathbb{X} \arrow[r, "\epsilon_1^{\mathbf{E}}"] & \mathbf{E}^1\mathbb{X} \arrow[r, "\epsilon_0^{\mathbf{E}}"] & \mathbf{E}^0\mathbb{X}
\end{tikzcd}$$

We denote this limit by $\mathbf{E}^{\omega}\mathbb{X}$.

Now let $\mathcal{M}^n = U_{\mathcal{H}}^{[n]}\, \mathrm{Mod}^{[n]} : \mathrm{GlobMult} \to \mathrm{GlobMult}$, and let $\epsilon_i^{\mathcal{H}} : \mathcal{M}_{i+1} \to \mathcal{M}_i$ be the transformation which forgets modules data at level n. Let $\mathcal{F}^{[n]} = U_{\amalg}^{[n]}\, \mathrm{Fam}^{[n]}\, U_{\amalg} : \mathrm{GlobMult}_{\amalg} \to \mathrm{GlobMult}$, and let $\epsilon_n^{\amalg} : \mathcal{F}^{n+1} \to \mathcal{F}^n$ be the transformation which computes coproducts at level $n+1$. Let

$$\Phi^n : \mathbf{E}^n \implies \mathcal{M}^n \mathcal{F}^n$$

be the natural isomorphism induced by Corollary 3.5.2.5. Then by naturality we have the following result:

Proposition 3.5.3.1. *The following square of natural transformations commutes:*

$$\begin{tikzcd}
\mathbf{E}^{n+1}U_{\amalg} \arrow[r, "\Phi_{n+1}"] \arrow[d, "\epsilon_n^{\mathbf{E}}"'] & \mathcal{M}^{n+1}\mathcal{F}^{n+1} \arrow[d, "\mathcal{M}^n \epsilon_n^{\amalg} \circ \epsilon_n^{\mathcal{H}} \mathcal{F}^{n+1}"] \\
\mathbf{E}^n U_{\amalg} \arrow[r, "\Phi_n"'] & \mathcal{M}^n \mathcal{F}^n
\end{tikzcd}$$

For any globular multicategory $\mathbb{Y}$, let $\mathcal{M}^{\omega}\mathbb{Y}$ be the limit of the following diagram in GlobMult:

$$\begin{tikzcd}
\cdots \arrow[r, "\epsilon_2^{\mathcal{H}}\mathbb{Y}"] & \mathcal{M}^2\mathbb{Y} \arrow[r, "\epsilon_1^{\mathcal{H}}\mathbb{Y}"] & \mathcal{M}^1\mathbb{Y} \arrow[r, "\epsilon_0^{\mathcal{H}}\mathbb{Y}"] & \mathcal{M}^0\mathbb{Y}
\end{tikzcd}$$

Theorem 3.5.3.2. *We have a natural isomorphism*

$$\Phi_\omega : \mathbf{E}^\omega \overset{\cong}{\Longrightarrow} \mathcal{M}^\omega \mathcal{F}^\omega \approx \mathrm{Mod}^{[\omega]}\,\mathrm{Fam}^{[\omega]}\,U_{\amalg}.$$

Proof. Recall that, for any globular multicategory $\mathbb{Y}$, $\mathcal{M}^\omega\mathbb{Y}$ is the limit of the following diagram in GlobMult:

$$\cdots \xrightarrow{\epsilon_2^{\mathcal{H}}\mathbb{Y}} \mathcal{M}^2\mathbb{Y} \xrightarrow{\epsilon_1^{\mathcal{H}}\mathbb{Y}} \mathcal{M}^1\mathbb{Y} \xrightarrow{\epsilon_0^{\mathcal{H}}\mathbb{Y}} \mathcal{M}^0\mathbb{Y}$$

Similarly, for any globular multicategory with coproducts $\mathbb{X}$, Remark 2.9.3.9 tells us that $\mathcal{F}^\omega\mathbb{X}$ is the equivalent to the limit of the following diagram in GlobMult:

$$\cdots \xrightarrow{\epsilon_2^{\amalg}\mathbb{X}} \mathcal{F}^2\mathbb{X} \xrightarrow{\epsilon_1^{\amalg}\mathbb{X}} \mathcal{F}^1\mathbb{X} \xrightarrow{\epsilon_0^{\amalg}\mathbb{X}} \mathcal{F}^0\mathbb{X}$$

Now consider the following diagram, whose squares commute by naturality:

$$\begin{tikzcd}
\cdots \arrow[r, "\epsilon_2^{\mathcal{H}}\mathcal{F}^0"] & \mathcal{M}^2\mathcal{F}^0\mathbb{X} \arrow[r, "\epsilon_1^{\mathcal{H}}\mathcal{F}^0"] & \mathcal{M}^1\mathcal{F}^0\mathbb{X} \arrow[r, "\epsilon_0^{\mathcal{H}}\mathcal{F}^0"] & \mathcal{M}^0\mathcal{F}^0\mathbb{X} \\
\cdots \arrow[u] \arrow[r, "\epsilon_2^{\mathcal{H}}\mathcal{F}^1"] & \mathcal{M}^2\mathcal{F}^1\mathbb{X} \arrow[u, "\mathcal{M}^2\epsilon_{\amalg}^0"] \arrow[r, "\epsilon_1^{\mathcal{H}}\mathcal{F}^1"] & \mathcal{M}^1\mathcal{F}^1\mathbb{X} \arrow[u, "\mathcal{M}^1\epsilon_{\amalg}^0"] \arrow[r, "\epsilon_0^{\mathcal{H}}\mathcal{F}^1"] & \mathcal{M}^0\mathcal{F}^1\mathbb{X} \arrow[u, "\mathcal{M}^0\epsilon_{\amalg}^0"] \\
\cdots \arrow[u] \arrow[r, "\epsilon_2^{\mathcal{H}}\mathcal{F}^2"] & \mathcal{M}^2\mathcal{F}^2\mathbb{X} \arrow[u, "\mathcal{M}^2\epsilon_{\amalg}^1"] \arrow[r, "\epsilon_1^{\mathcal{H}}\mathcal{F}^2"] & \mathcal{M}^1\mathcal{F}^2\mathbb{X} \arrow[u, "\mathcal{M}^1\epsilon_{\amalg}^1"] \arrow[r, "\epsilon_0^{\mathcal{H}}\mathcal{F}^2"] & \mathcal{M}^0\mathcal{F}^2\mathbb{X} \arrow[u, "\mathcal{M}^0\epsilon_{\amalg}^1"] \\
\cdots \arrow[u] \arrow[r] & \vdots \arrow[u, "\mathcal{M}^2\epsilon_{\amalg}^2"] \arrow[r] & \vdots \arrow[u, "\mathcal{M}^1\epsilon_{\amalg}^2"] \arrow[r] & \vdots \arrow[u, "\mathcal{M}^0\epsilon_{\amalg}^2"]
\end{tikzcd}$$

Since each $\mathcal{M}_n$ preserves limits, taking the limit of the rows we obtain

$$\cdots \xrightarrow{\epsilon_2^{\mathcal{H}}\mathcal{F}^\omega} \mathcal{M}^2\mathcal{F}^\omega\mathbb{X} \xrightarrow{\epsilon_1^{\mathcal{H}}\mathcal{F}^\omega} \mathcal{M}^1\mathcal{F}^\omega\mathbb{X} \xrightarrow{\epsilon_0^{\mathcal{H}}\mathcal{F}^\omega} \mathcal{M}^0\mathcal{F}^\omega\mathbb{X}$$

Consequently, the limit of the whole square is the limit of this diagram, and this is $\mathcal{M}^\omega\mathcal{F}^\omega\mathbb{X}$. On the other hand, the limit of the whole square must be the limit of the diagonal,

$$\cdots \xrightarrow{\mathcal{M}^2\epsilon_{\amalg}^2\circ\epsilon_2\mathcal{F}^2} \mathcal{M}^2\mathcal{F}^2 \xrightarrow{\mathcal{M}^1\epsilon_{\amalg}^1\circ\epsilon_1\mathcal{F}^1} \mathcal{M}^1\mathcal{F}^1 \xrightarrow{\mathcal{M}^0\epsilon_{\amalg}^0\circ\epsilon_0\mathcal{F}^0} \mathcal{M}^0\mathcal{F}^0.$$

Applying, Proposition 3.5.3.1, this diagram is isomorphic to

$$\cdots \xrightarrow{\epsilon_2^{\mathbf{E}}} \mathbf{E}^2\mathbb{X} \xrightarrow{\epsilon_1^{\mathbf{E}}} \mathbf{E}^1\mathbb{X} \xrightarrow{\epsilon_0^{\mathbf{E}}} \mathbf{E}^0\mathbb{X},$$

and the limit of this diagram is $\mathbf{E}^\omega\mathbb{X}$ by definition. Hence, we have constructed a natural isomorphism $\Phi_\omega : \mathbf{E}^\omega \cong \mathcal{M}^\omega\mathcal{F}^\omega$ as required. □

Example 3.5.3.3. Suppose that $\mathbb{X} = \mathbb{1}$. Then $\mathbf{E}^\omega \mathbb{1}$ is the limit of the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\epsilon^{\mathbf{E}}_1(\mathbb{1})"] & \mathbf{E}^1\mathbb{1} \arrow[r, "\epsilon^{\mathbf{E}}_0(\mathbb{1})"] & \mathbf{E}^0\mathbb{1} \arrow[r, "!"] & \mathbb{1}
\end{tikzcd}$$

However, Theorem 3.5.3.2 implies that $\mathbf{E}^\omega \mathbb{1} \approx \mathrm{Mod}\,\mathrm{SpanSet}$.Thus, in the spirit of [20], we have exhibited Mod SpanSet as the canonical limit of a process of iterated enrichment. In fact, we can make a precise comparison with the description of $\mathrm{Str}\,\omega$-Cat found ibid. Let $\mathbb{1}_{\mathsf{Sym}}$ be the terminal symmetric monoidal globular multicategory. Then consider the limit $\mathbf{E}^\omega \mathbb{1}_{\mathsf{Sym}}$ of the following diagram in $\mathrm{GlobMult}_{\mathsf{Sym}}$:

$$\begin{tikzcd}
\cdots \arrow[r, "\mathbf{E}^2!"] & \mathbf{E}^2\mathbb{1}_{\mathsf{Sym}} \arrow[r, "\mathbf{E}!"] & \mathbf{E}\mathbb{1}_{\mathsf{Sym}} \arrow[r, "!"] & \mathbb{1}_{\mathsf{Sym}},
\end{tikzcd} \tag{$\dagger$}$$

Applying the functor $(-)_0 : \mathrm{GlobMult}_{\mathsf{Sym}} \to \mathrm{GlobMult}$, we obtain precisely the above globular multicategory, and taking the limit we find that $(\mathbf{E}^\omega \mathbb{1}_{\mathsf{Sym}})_0 = \mathbf{E}^\omega \mathbb{1}$. Thus,

$$(\mathbf{E}^\omega \mathbb{1}_{\mathsf{Sym}})_0 = \mathbf{E}^\omega \mathbb{1} \approx \mathrm{Mod}\,\mathrm{SpanSet}$$

On the other hand, since $\mathcal{V}_\otimes \mathbb{1} = \top$, we have an isomorphism of diagrams:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d, "\cong"] & \mathcal{V}_\otimes \mathbf{E}^2\mathbb{1}_{\mathsf{Sym}} \arrow[r] \arrow[d, "\cong"] & \mathcal{V}_\otimes \mathbf{E}^1\mathbb{1}_{\mathsf{Sym}} \arrow[r] \arrow[d, "\cong"] & \mathcal{V}_\otimes \mathbf{E}^0\mathbb{1}_{\mathsf{Sym}} \arrow[d, "\cong"] \\
\cdots \arrow[r] & \mathbf{E}^2\top \arrow[r] & \mathbf{E}^1\top \arrow[r] & \mathbf{E}^0\top
\end{tikzcd}$$

This induces an isomorphism between the limits of the top and bottom rows in SymMonCat. By the results of [20], the limit of the bottom row is the symmetric monoidal category of strict ω-omega categories. Since $\mathcal{V}_\otimes$ is a right adjoint, it preserves limits. Hence, the limit of the top row is $\mathcal{V}_\otimes \mathbf{E}^\omega \mathbb{1}_{\mathsf{Sym}}$. Thus,

$$\mathcal{V}_\otimes \mathbf{E}^\omega \mathbb{1}_{\mathsf{Sym}} \approx \mathrm{Str}\,\omega\text{-Cat}.$$

Example 3.5.3.4. Both $\mathrm{Mod}^{\{i\}}$ and $\mathrm{Fam}^{\{i\}}$ preserve limits of ω-length chains. It follows that $\mathbf{E}$ preserves limits of these chains. Hence $\mathbf{E}\mathbf{E}^\omega \mathbb{1}_{\mathsf{Sym}} \cong \mathbf{E}^\omega \mathbb{1}_{\mathsf{Sym}}$. Combing this observation with Remark 3.5.1.2, we have that

$$\begin{aligned}
\mathrm{tr}_1\,\mathrm{Mod}\,\mathrm{SpanSet} &\approx \mathrm{tr}_1(\mathbf{E}^\omega \mathbb{1}_{\mathsf{Sym}})_0 \\
&\cong \mathrm{tr}_1(\mathbf{E}\mathbf{E}^\omega \mathbb{1}_{\mathsf{Sym}})_0 \\
&\cong \mathrm{tr}_1\mathbf{E}_0(\mathbf{E}^\omega \mathbb{1}_{\mathsf{Sym}})_1 \\
&\cong \mathbf{E}_0\mathrm{tr}_1(\mathbf{E}^\omega \mathbb{1}_{\mathsf{Sym}})_1
\end{aligned}$$

However, considering the definition of $\mathcal{V}_\otimes$ in Example 3.5.1.8, for any symmetric monoidal globular multicategory $\mathbb{X}$, we have that $\operatorname{tr}_1(\mathbb{X})_1 = \operatorname{Sq}\mathcal{V}_\otimes\mathbb{X}$. (Here we view $\mathcal{V}_\otimes$ as a 1-object 2-category.) Hence,

$$\operatorname{tr}_1 \operatorname{Mod} \operatorname{SpanSet} \approx \mathbf{E}_0 \operatorname{Sq} \mathcal{V}_\otimes \mathbf{E}^\omega \mathbb{1}_{\mathrm{Sym}} \approx E_0 \operatorname{Sq} \operatorname{Str} \omega\text{-Cat}.$$

Hence, by Example 3.5.1.5, we have that $\operatorname{tr}_1 \operatorname{Mod} \operatorname{SpanSet}$ is equivalent to the virtual double category of categories, functors, profunctors, and transformations enriched in strict ω-categories described by [17].

Chapter 4

Fibrational and Weak Homomorphism Types

In this chapter we weaken the rules defining strict homomorphism types in order to define two different notions of homomorphism type: *fibrational homomorphism types*, and *weak homomorphism types*. While types in globular multicategories with strict homomorphism types behave like strict higher categories, types in globular multicategories with weak or fibrational homomorphism types behave like weak higher categories. We show how models of dependent type theory with identity types induce globular multicategories with fibrational homomorphism types, while models of type theory with path types (that is propositional identity types) induce globular multicategories with weak homomorphism types.

4.1 Pre-homomorphism Types

We first describe a common structure underlying globular multicategories with strict, fibrational, and weak homomorphism types.

Definition 4.1.0.1. We say that a globular multicategory has *pre-homomorphism types* when its underlying globular multigraph is reflexive and we have the following structure:

- For each n-term $f : \Gamma \to M$, $sf \nrightarrow tf$ and each $(n-1)$-variable $x : A$ in Γ a *J-term*
$$J_x(f) : \Gamma \oplus_x \mathcal{H}_A \longrightarrow M, \quad sf \longrightarrow\!\!\!\!\!\mapsto tf.$$
- Suppose that $0 < k < n$. Let $f : \Gamma \to M$ be an n-term. Then for each k-variable

$x : A$ in Γ and any term-wise parallel $(n-1)$-terms

$$j_s : s\Gamma \oplus_x \mathcal{H}_A \longrightarrow sM$$
$$j_t : t\Gamma \oplus_x \mathcal{H}_A \longrightarrow tM$$

such that

$$\mathfrak{r}_x ; j_s = sf$$
$$\mathfrak{r}_x ; j_t = tf$$

we have a *J-term*

$$J_x^{j_s,j_t}(f) : \Gamma \oplus_x \mathcal{H}_A \longrightarrow M, \quad j_s \twoheadrightarrow j_t.$$

We denote the category of globular multicategories with pre-homomorphism types by $\mathrm{GlobMult}_{\mathcal{H}}^{\mathrm{Pre}}$.

Example 4.1.0.2. Every globular multicategory with strict homomorphism types has pre-homomorphism types.

Remark 4.1.0.3. An *algebraic pre-equivalence* of globular sets is a map of globular sets, $f : X \to Y$, together with:

- For each 0-cell $a \in Y(0)$, a 0-cell $\mathbf{j}_f(a) \in X(0)$.
- For each $n \geq 0$, each pair of parallel n-cells $a, b \in X(n)$, and each $(n+1)$-cell $c \in Y(n+1)$, such that $sc = f(a)$ and $tc = f(b)$, an $(n+1)$-cell $\mathbf{j}_f^{a,b}(c)$ such that $s\mathbf{j}^{a,b}(c) = a$ and $t\mathbf{j}^{a,b}(c) = b$.

Let $\mathbb{X}$ be a reflexive globular multicategory. Suppose that Γ is an n-context, A is an n-type A, $g : s\Gamma \to sA$ and $h : t\Gamma \to tA$ are term-wise parallel $(n-1)$-terms. Then, we define the globular set

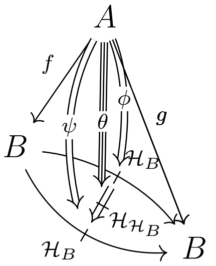
$$[[\Gamma \longrightarrow M, \quad g \twoheadrightarrow h]]$$

so that

$$\begin{aligned}
&[[\Gamma \longrightarrow M, \quad g \twoheadrightarrow h]](0) = [\Gamma \longrightarrow M, \quad g \twoheadrightarrow h],\\
&[[\Gamma \longrightarrow M, \quad g \twoheadrightarrow h]](1) = \{f : \Gamma \longrightarrow \mathcal{H}_M \mid s_{n-1}f = g,\, t_{n-1}f = h\},\\
&[[\Gamma \longrightarrow M, \quad g \twoheadrightarrow h]](2) = \{f : \Gamma \to \mathcal{H}_M^2 \mid s_{n-1}f = g,\, t_{n-1}f = h\},\\
&\qquad\qquad\qquad\qquad\vdots
\end{aligned}$$

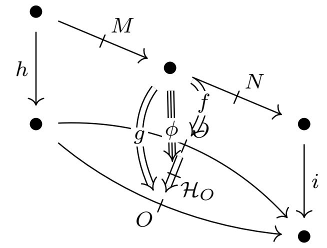
Example 4.1.0.4. Suppose that A and B are 0-types. Then, the following three diagrams depict a typical 0-cell $f \in [[A \to B,\, \star \nrightarrow \star]]$, a 1-cell ϕ such that $s\phi = f$ and $t\phi = g$, and a 2-cell θ such that $s\theta = \phi$ and $t\theta = \psi$ respectively:

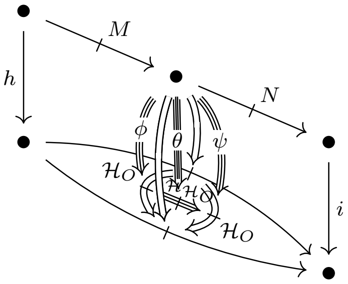
$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, Rightarrow, "\phi"] & \\ B \arrow[rr, "\mathcal{H}_B"', "|" marking] & {} & B \end{tikzcd}$$



Example 4.1.0.5. Suppose that M, N and O are 1-types. Then, the following three diagrams depict a typical 0-cell $f \in [[M \odot_0 N \to O,\, h \nrightarrow i]]$, a 1-cell ϕ such that $s\phi = f$ and $t\phi = g$, and a 2-cell θ such that $s\theta = \phi$ and $t\theta = \psi$ respectively:

$$\begin{tikzcd} \bullet \arrow[r, "M", "|" marking] \arrow[d, "h"'] & \bullet \arrow[r, "N", "|" marking] \arrow[d, Rightarrow, "f"] & \bullet \arrow[d, "i"] \\ \bullet \arrow[rr, "O"', "|" marking] & {} & \bullet \end{tikzcd}$$





Each n-substitution $r : \Delta \to \Gamma$ induces a composition map

$$[[\Gamma \longrightarrow A, \quad g \nrightarrow h]] \xrightarrow{r;-} [[\Delta \longrightarrow A, \quad sr; g \nrightarrow tr; h]].$$

Let S be a set of types in $\mathbb{X}$ such that, for each m-type in S, we have that $m \geq n$. To give a *pre-representation* structure on r *relative to* S is to give, for each $m \geq n$, each m-type M in S, and each pair of term-wise parallel $(m-1)$-terms $g : s\Gamma \to sM$, $h : t\Gamma \to tM$, a choice of algebraic pre-equivalence structure on the map $r; -$. When S is the set of all m-types in $\mathbb{X}$, for $m \geq n$, we omit the “relative to” part of this definition. When $S = \{M\}$ is a singleton, we will speak of (pre)-representations relative to M.

It follows that to equip a reflexive globular multicategory with pre-homomorphism types is to give, for each $k < n$, each n-context Γ, and each k-variable $x : A$ in Γ, a pre-representation structure for the reflexivity substitution $\mathfrak{r}^{\Gamma}_{x} : \Gamma \oplus_x \mathcal{H}_A \to \Gamma$.

4.1.1 ω-Precategories

Globular multicategories with pre-homomorphism types already have enough structure to endow collections of terms with notions composition and unit. However, this data need not be coherent. We follow [13] and make the following definition:

Definition 4.1.1.1. An *ω-precategory* is a globular set X together with

- A composition operation $- \circ - : X(n) \times_{X(n-1)} X(n) \to X(n)$, allowing us to compose parallel n-terms, such that whenever $\phi, \psi \in X_n$ and $t\phi = s\psi$, we have that $s(\phi \circ \psi) = s\phi$ and $t(\phi \circ \psi) = t\psi$.
- An identity operation $\mathrm{id} : X(n) \to X(n+1)$ such that $s \,\mathrm{id}_f = t \,\mathrm{id}_f = f$

A *homomorphism* of ω-precategories is a map of globular sets preserving the composition and identity operations.

Definition 4.1.1.2. Suppose that f, g are parallel n-cells in a globular set X. Then a *transformation* $\phi : f \to g$ is an $(n+1)$-cell ϕ in X such that $s\phi = f$, and $t\phi = g$.

Suppose that $\mathbb{X}$ is a reflexive globular multicategory. Suppose that $f, g : \Gamma \to A,\ sf \nrightarrow tf$ are parallel n-terms in $\mathbb{X}$. Then a *transformation* $\phi : f \to g$ is a term $\phi : \Gamma \to \mathcal{H}_A,\ f \nrightarrow g$; that is, a 1-cell in $[[\Gamma \to A,\ sf \nrightarrow tf]]$.

Definition 4.1.1.3. Suppose that $\mathbb{X}$ is a globular multicategory with pre-homomorphism types. Suppose that M is an n-type in $\mathbb{X}$, and that $\mathcal{H}_x : \mathcal{H}_A$ is in the canonical n-variable in the n-context $[\mathcal{H}_A]$. Suppose that $x_1 = t\mathcal{H}_x$. Then we define

$$\mathfrak{m}_A = \mathfrak{J}_{x_1}(\mathrm{id}_{\mathcal{H}_A}) : \mathcal{H}_A \odot_n \mathcal{H}_A \longrightarrow \mathcal{H}_A, \quad \mathrm{id}_A \nrightarrow \mathrm{id}_A\,.$$

When $M : A \nrightarrow B$ is an n-type, and $m : M$ is the unique n-variable in $[M]$, we define

$$\lambda_M = \mathfrak{J}_{sm}(\mathrm{id}_M), \qquad \rho_M = \mathfrak{J}_{tm}(\mathrm{id}_M).$$

Definition 4.1.1.4. Suppose that $\phi : f \to g$ and $\psi : g \to h$ are transformations between n-terms in globular multicategory with pre-homomorphism types $\mathbb{X}$. Then we define the composite $\phi \circ \psi : f \to h$ by

$$\phi \circ \psi = (\phi \odot_n \psi); \mathfrak{m}_A.$$

We define the unit id_f of a term $f : \Gamma \to A,\ sf \nrightarrow tf$ by

$$\mathrm{id}_f = f; \mathfrak{r}_A.$$

Thus, we have equipped $[[\Gamma \to A,\ sf \nrightarrow tf]]$ with the structure of an ω-precategory.

Definition 4.1.1.5. Suppose that $\phi : f \to f'$. Then, given a term $g : \Gamma \to M, f' \nrightarrow tg$, we define the composite

$$\phi \circ g : f \nrightarrow tg$$

to be $(\phi \odot_n g); \lambda_M$. Similarly, given $h : \Gamma \to M$, $sh \nrightarrow f$, we define

$$h \circ \phi : sh \nrightarrow f'$$

to be $(g \odot_n \phi); \rho_M$.

Example 4.1.1.6. Suppose that $a : \Delta \to \Gamma$, $sa \nrightarrow ta$ is a substitution in $\mathbb{X}$. Then it follows that the composition operation

$$[[\Gamma \longrightarrow A, \quad g \nrightarrow h]] \xrightarrow{a;-} [[\Delta \longrightarrow A, \quad sa; g \nrightarrow ta; h]].$$

is a homomorphism of ω-precategories

This ω-precategory structure allows us to define a notion of *equivalence* between n-terms in $\mathbb{X}$. Indeed, there are a number of good candidates for the notion of equivalence. See [13] where two different notions are compared, and see also [53][§4] where the similar problem of defining equivalences in homotopy type theory is considered. We will adapt the *bi-invertible* maps of the latter source to our setting. This notion is studied in [39].

Definition 4.1.1.7. We define equivalences between parallel n-cells coinductively. Suppose that f, g are parallel n-cells in an ω-precategory. Then an *equivalence*

$$\phi : f \approx g$$

consists of a transformation $\phi : f \to g$, together with a pair of transformations $\phi^L, \phi^R : g \to f$, and equivalences between $(n+1)$-terms:

$$\phi^L \circ \phi \approx \mathrm{id}_g, \qquad \phi \circ \phi^R \approx \mathrm{id}_f \,.$$

Proposition 4.1.1.8. *Whenever $F : X \to Y$ is a homomorphism of ω-precategories, and $f, g \in X(k)$ are parallel cells, we have that $f \approx g \implies F(f) \approx F(g)$.*

Proof. This follows from a straightforward coinduction. □

Example 4.1.1.9. Suppose that $f \approx g : \Gamma \to A$ are equivalent n-terms in a globular multicategory with pre-homomorphism types. Suppose that $a : \Delta \to \Gamma$ is an n-substitution. Then by Proposition 4.1.1.8 applied to $a; -$, we have that $a; f \approx a; g$.

Definition 4.1.1.10. We say that an ω-precategory is *idempotent* if, for any n-cell f, we have that

$$\mathrm{id}_f \circ \mathrm{id}_f \approx \mathrm{id}_f \,.$$

Example 4.1.1.11. Suppose that X is an idempotent ω-precategory, and that $f \in X(k)$. Then, since,

$$\mathrm{id}_f \circ \mathrm{id}_f \approx \mathrm{id}_f$$

we have that $f \approx f$.

Definition 4.1.1.12. We say that a globular multicategory $\mathbb{X}$ has *idempotent* pre-homomorphism types when, for any n-type A, we have that

$$\mathfrak{r}_A \circ \mathfrak{r}_A \approx \mathfrak{r}_A.$$

Remark 4.1.1.13. Since for any n-term f, we have that $\mathrm{id}_f = f; \mathfrak{r}_A$, pre-homomorphism types are idempotent exactly when, for all f,

$$\mathrm{id}_f \circ \mathrm{id}_f \approx \mathrm{id}_f\,.$$

Thus, $\mathbb{X}$ has idempotent pre-homomorphism types if and only if for any n-context Γ, any n-type A, and any term-wise parallel $(n-1)$-terms $g : s\Gamma \to sA$ and $h : t\Gamma \to tA$, the ω-precategory.

$$[[\Gamma \longrightarrow A, \quad g \twoheadrightarrow h]]$$

is idempotent.

Definition 4.1.1.14. A pre-equivalence between ω-precategories is a homomorphism with the structure of a pre-equivalence between their underlying globular sets.

Remark 4.1.1.15. We say that an pre-equivalence is an *algebraic acyclic fibration* when additionally:

- For each 0-cell $a \in Y(0)$, we have that

$$f(\mathbf{j}(a)) = a,$$

- For each $n \geq 0$, each pair of parallel n-cells $g, h \in X(n)$, and each $(n+1)$-cell $c \in Y(n+1)$, such that $sc = f(g)$ and $tc = f(h)$, we have that

$$f(\mathbf{j}^{g,h}(c)) = c.$$

If, furthermore, the choices defining an algebraic acyclic fibration are unique, then it follows that f is a level-wise bijection; that is, an isomorphism of globular sets.

Definition 4.1.1.16. An *algebraic weak equivalence* of ω-precategories is a pre-equivalence $F : X \to Y$ together with the following data:

- For each 0-cell f in $Y(0)$, an equivalence $F(\mathbf{j}(f)) \approx f$
- For each $n > 0$, each parallel $(n-1)$-cells $g, h \in X(n-1)$, and each $f \in Y(n)$ such that $F(g) = sf$ and $F(h) = tf$, an equivalence

$$\nu_F^{g,h}(f) : F(\mathbf{j}_F^{g,h}(f)) \approx f$$

We say that a homomorphism $F : X \to Y$ is a *weak equivalence* when it can be equipped with the structure of an algebraic weak equivalence.

Thus, a homomorphism of ω-precategories is a weak equivalence when it is, in a precise sense, essentially surjective at all levels. The following propositions immediately follow from this definition:

Example 4.1.1.17. Whenever X is idempotent, Example 4.1.1.11 implies that the identity homomorphism $\mathrm{id}_X : X \to X$ is a weak equivalence.

Proposition 4.1.1.18. *Weak equivalences are closed under composition.*

Proof. This follows from Proposition 4.1.1.8. □

Proposition 4.1.1.19. *Whenever $F : X \to Y$ is a weak equivalence and $F(f) \approx F(g)$ in Y, we have that $f \approx g$ in X.*

Proof. Suppose that we have an equivalence between n-cells $\phi : F(f) \approx F(g)$. Then we have

$$\mathbf{j}_F^{f,g}(\phi) : f \longrightarrow g, \qquad \mathbf{j}_F^{g,f}(\phi^L) : g \longrightarrow f, \qquad \mathbf{j}_F^{g,f}(\phi^R) : g \longrightarrow f.$$

We have that

$$\begin{aligned} F(\mathbf{j}_F^{g,f}(\phi^L) \circ \mathbf{j}_F^{f,g}(\phi)) &= F(\mathbf{j}_F^{g,f}(\phi^L)) \circ F(\mathbf{j}_F^{f,g}(\phi)) \\ &= \phi^L \circ \phi \\ &\approx \mathrm{id}_g, \end{aligned}$$

and, by a similar argument, $F(\mathbf{j}_F^{g,f}(\phi) \circ \mathbf{j}_F^{f,g}(\phi^R)) \approx \mathrm{id}_f$. Since these are equivalences of $(n+1)$-cells in $\mathbb{Y}$, the result now follows by coinduction. □

4.2 Definition

We now describe two weakly coherent notions of homomorphism type.

Definition 4.2.0.1. We say that a globular multicategory has *fibrational homomorphism types* when it has pre-homomorphism types such that:

- For each n-term $f : \Gamma \to A$ and each $(n-1)$-variable x, we have that
$$\mathfrak{r}_x; J_x(f) = f.$$
- Suppose that $0 < k < n$. Let $f : \Gamma \to M$ be an n-term. Then for each k-variable $x : A$ in Γ and any term-wise parallel $g : s\Gamma \oplus_x \mathcal{H}_A \to sM$, $h : t\Gamma \oplus_x \mathcal{H}_A \to tM$ such that $\mathfrak{r}_x; g = sf, \mathfrak{r}_x; h = tf$, we have that
$$\mathfrak{r}_x; J_x^{g,h}(f) = f.$$

We denote the category of globular multicategories with fibrational homomorphism types by $\mathrm{GlobMult}_{\mathcal{H}}^{\mathrm{Fib}}$.

Definition 4.2.0.2. A *globular multicategory with weak homomorphism types* is a globular multicategory with pre-homomorphism types, together with the following data:

- For each n-term $f : \Gamma \to A$ and each $(n-1)$-variable x, we require an equivalence
$$\nu_x(f) : \mathfrak{r}_x; J_x(f) \approx f.$$
- Suppose that $0 < k < n$. Let $f : \Gamma \to M$ be an n-term. Then, for each k-variable $x : A$ in Γ and any term-wise parallel $g : s\Gamma \oplus_x \mathcal{H}_A \to sM, h : t\Gamma \oplus_x \mathcal{H}_A \to tM$ such that $\mathfrak{r}_x; g = sf, \mathfrak{r}_x; h = tf$, we require an equivalence
$$\nu_x^{g,h}(f) : \mathfrak{r}_x; J_x^{g,h}(f) \approx f.$$

We denote the category of globular multicategories with weak homomorphism types by $\mathrm{GlobMult}_{\mathcal{H}}^{\mathrm{Wk}}$.

Remark 4.2.0.3. Suppose that $r : \Delta \to \Gamma$ is a substitution in a globular multicategory with pre-homomorphism types $\mathbb{X}$. Let S be a set of types in $\mathbb{X}$ such that, for each m-type in S, $m \geq n$. Recall that a pre-representation structure on r relative to S is a choice, for each $m \geq n$, each m-type M in S, and each pair of term-wise parallel $(m-1)$-terms $g : s\Gamma \to sM, h : t\Gamma \to tM$, of an algebraic pre-equivalence structure on the map

$$[[\Gamma \longrightarrow M, \quad g \twoheadrightarrow h]] \xrightarrow{r;-} [[\Delta \longrightarrow M, \quad sr;g \twoheadrightarrow tr;h]].$$

A *fibrational representation structure on r relative to S* is a pre-representation structure on r whose pre-equivalences are algebraic acyclic fibrations. A *weak representation* structure on r relative to S is data making these pre-equivalences algebraic weak equivalences.

Recall that to equip a reflexive globular multicategory with pre-homomorphism types is to give, for each $k < n$, each n-context Γ, and each k-variable $x : A$ in Γ, a pre-representation structure for the reflexivity substitution $\mathfrak{r}_x^\Gamma : \Gamma \oplus_x \mathcal{H}_A \to \Gamma$. It follows that a choice of pre-homomorphism types is a choice of fibrational homomorphism types if these pre-representations are fibrational representations. Furthermore, to give a choice of pre-homomorphism types the structure of weak homomorphism types is to choose data making these pre-representations weak representations.

Remark 4.2.0.4. We say that a fibrational representation on r is *strict*, when the choices defining the associated algebraic acyclic fibrations are unique. In this case, each map

$$[[\Gamma \longrightarrow A, \quad g \twoheadrightarrow h]] \xrightarrow{r;-} [[\Delta \longrightarrow A, \quad sr;g \twoheadrightarrow tr;h]].$$

is an isomorphism of globular sets and, furthermore, the pre-equivalence $\mathbf{j}$ defines the inverse of this map. It follows that a substitution has a (necessarily unique) strict representation structure if and only if it is strictly representing.

4.2.1 Examples

Example 4.2.1.1. Strict homomorphism types are fibrational homomorphism types. In fact, a globular multicategory with fibrational homomorphism types has strict homomorphism types if and only if for any parallel terms f and f' we have that

$$\mathfrak{r}_x; f = \mathfrak{r}_x; f' \implies f = f'.$$

Example 4.2.1.2. Suppose that $\mathbb{X}$ has fibrational homomorphism types. Then, $\mathbb{X}$ has weak homomorphism types.

Example 4.2.1.3. Recall that every type theory, $\mathcal{T}$, induces a globular multicategory $\mathbb{G}_T(\mathcal{T})$. When $\mathcal{T}$ is equipped with identity types, the identity types, reflexivity terms and the computation rules for these types in $\mathcal{T}$ equip the globular $\mathbb{G}_T(\mathcal{T})$ with fibrational homomorphism types. When $\mathcal{T}$ is equipped with path types, the path types, reflexivity terms and the computation rules for these types in $\mathcal{T}$ equip the globular $\mathbb{G}_T(\mathcal{T})$ with weak homomorphism types. We will make this precise in Section 4.3 below.

Example 4.2.1.4. Similar definitions work for the finite case. In the 1-dimensional case strict, fibrational, and weak homomorphism types all coincide.

Example 4.2.1.5. Suppose that $\mathbb{P}$ is a globular operad with a choice of contraction. Then $\mathbb{P}$ can be equipped with pre-homomorphism types as follows:

- For each type n, we define $\mathcal{H}_n = n+1$. We define $\mathfrak{r}_n = \mathbf{l}_n^{\mathrm{id}^n,\mathrm{id}^n}$.

- For each n-term $f : \pi \to n$, $sf \nrightarrow tf$, and each variable $x : A \in \pi(n-1)$, we define
$$\mathfrak{J}_x(f) = \mathbf{l}_{\pi \oplus_x \mathcal{H}_A}^{sf,st} : \pi \oplus_x \mathcal{H}_A \longrightarrow n, \quad sf \nrightarrow tf.$$
  Note that $\mathfrak{r}_x ; \mathfrak{J}_x(f) : \pi \to n$, $sf \nrightarrow tf$ is parallel to f.

- For each $k < n-1$, each n-term $f : \pi \to n$, $sf \nrightarrow tf$, each variable $x \in \pi(k)$, and each pair of term-wise parallel $(n-1)$-terms $g : \pi_\partial \oplus_x \mathcal{H}_A \to n-1$ and $h : \pi_\partial \oplus_x \mathcal{H}_A \to n-1$ such that $\mathfrak{r}_x ; g = sf$ and $\mathfrak{r}_x ; h = tf$, we define
$$\mathfrak{J}_x^{g,h}(f) = \mathbf{l}_{\pi \oplus_x \mathcal{H}_A}^{g,h} : \pi \oplus_x \mathcal{H}_A \longrightarrow n, \quad g \nrightarrow h.$$
  Note that $\mathfrak{r}_x ; \mathfrak{J}_x^{g,h}(f) : \pi \to n$, $sf \nrightarrow tf$ is parallel to f.

These data induce weak homomorphism types by the Lemma 4.2.1.6 below.

Lemma 4.2.1.6. *Suppose that $\mathbb{P}$ is a globular operad with a choice of contraction. Suppose that $f, g : \pi \to n$ are parallel n-terms in $\mathbb{P}$. Then $f \approx g$.*

Proof. We define $\phi : f \to g$, and $\psi : g \to f$ by
$$\phi = \mathbf{l}_\pi^{f,g}, \qquad \psi = \mathbf{l}^{g,f}.$$
Then the $(n+1)$-term $\psi \circ \phi : \pi \longrightarrow n+1, \quad g \nrightarrow g$ is parallel to id_g. Similarly, the $(n+1)$-term $\phi \circ \psi$ is parallel to id_f. Hence, the result follows by coinduction. □

Proposition 4.2.1.7. *The forgetful functor* $U_{\mathcal{H}} : \mathrm{GlobMult}^{\mathrm{Wk}}_{\mathcal{H}} \to \mathrm{GlobMult}$ *creates pullbacks.*

Proof. Suppose that we have a pullback diagram

$$\begin{tikzcd}
\mathbb{X} \times_{\mathbb{Z}} \mathbb{Y} \arrow[r, "\pi_{\mathbb{X}}"] \arrow[d, "\pi_{\mathbb{Y}}"'] & \mathbb{X} \arrow[d, "\mathbb{F}"] \\
\mathbb{Y} \arrow[r, "\mathbb{G}"'] & \mathbb{Z}
\end{tikzcd}$$

in GlobMult such that $\mathbb{X}, \mathbb{Y}$ and $\mathbb{Z}$ are globular multicategories with weak homomorphism types, and $\mathbb{F}$ and $\mathbb{G}$ preserve this data. Then to define a type (or term) A in $\mathbb{X} \times_{\mathbb{Z}} \mathbb{Y}$ is to define a type (or term) $\pi_{\mathbb{X}} A$ in $\mathbb{X}$, and a type (or term) $\pi_{\mathbb{Y}} A$ in $\mathbb{Y}$ such that $\mathbb{F}\pi_{\mathbb{X}} A = \mathbb{G}\pi_{\mathbb{Y}} A$. Since $\mathbb{F}$ and $\mathbb{G}$ preserve weak homomorphism types, this allows us to construct weak homomorphism types for $\mathbb{X} \times_{\mathbb{Z}} \mathbb{Y}$. For example, for each type A in $\mathbb{X} \times_{\mathbb{Z}} \mathbb{Y}$. we define $\mathcal{H}_A : A \nrightarrow A$ to be the unique type such that $\pi_{\mathbb{X}} \mathcal{H}_A = \mathcal{H}_{\pi_{\mathbb{X}} A}$, and $\pi_{\mathbb{Y}} \mathcal{H}_A = \mathcal{H}_{\pi_{\mathbb{Y}} A}$. In fact, given this choice of weak homomorphism types, $\pi_{\mathbb{X}}$ and $\pi_{\mathbb{Y}}$ preserve weak homomorphism types. This property uniquely characterises this choice of weak homomorphism types. Furthermore, the corresponding square in $\mathrm{GlobMult}^{\mathrm{Wk}}_{\mathcal{H}}$ is clearly a pullback square. □

Corollary 4.2.1.8. *Discrete opfibrations reflect weak homomorphism types.*

Proof. Recall that the globular multicategory of pointed sets $\mathrm{SpanSet}_\star$ has strict homomorphism types, and that the universal discrete opfibration $\pi_\star : \mathrm{SpanSet}_\star \to \mathrm{SpanSet}$ preserves strict homomorphism types. Up to size constraints, every discrete opfibration can be described as a pullback $\pi_{\mathbb{F}}$ as in the diagram below:

$$\begin{tikzcd}
\mathrm{el}(\mathbb{F}) \arrow[r] \arrow[d, "\pi_{\mathbb{F}}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{SpanSet}_\star \arrow[d, "\pi_\star"] \\
\mathbb{X} \arrow[r, "\mathbb{F}"'] & \mathrm{SpanSet}
\end{tikzcd}$$

Suppose that $\mathbb{X}$ has weak homomorphism types. Then Proposition 4.2.1.7 implies that $\mathrm{el}(\mathbb{F})$ has weak homomorphism types, and that $\pi_{\mathbb{F}}$ preserves this data. □

Corollary 4.2.1.9. *Whenever* $\mathcal{C} : \mathbb{P} \to \mathrm{SpanSet}$ *is a weak higher category parametrized by a contractible globular operad, the globular multicategory of elements* $\mathrm{el}\,\mathcal{C}$ *has weak homomorphism types.*

Proposition 4.2.1.10. *Suppose that* $\mathbb{P}$ *is a normalised contractible globular operad with strict composition along* 0*-types, with a choice of contraction. Suppose that* $\mathcal{C} : \mathbb{P} \to \mathrm{SpanSet}$ *is a weak higher category parametrized by* $\mathbb{P}$*. Then, the vertical globular multicategory* $\mathbb{V}(\mathcal{C})$ *has weak homomorphism types.*

Proof. Choose weak homomorphism types for $\mathbb{P}$ using Example 4.2.1.5. For each n-type $\mathcal{H}_A^n$ in $\mathbb{V}(\mathcal{C})$, we define $\mathcal{H}_{\mathcal{H}_A^n} = \mathcal{H}_A^{n+1}$. We define $\mathfrak{r}_{\mathcal{H}_A^n} : \mathcal{H}_A^n \to \mathcal{H}_A^{n+1}$ so that

$$\overline{\mathfrak{r}_{\mathcal{H}_A^n}} = \mathbb{F}(c_0^{n+2})(\bar{A}),$$
$$\mathbf{o}_{\mathfrak{r}_{\mathcal{H}_A^n}} = \mathfrak{r}_{n+1} : n+1 \longrightarrow n+2,\ \mathrm{id}_{n+1} \twoheadrightarrow \mathrm{id}_{n+1}\,.$$

Suppose that π is an n-pasting diagram, and that $x \in \pi(k)$ for some $k < n$. Then $\Sigma(\pi \oplus_x \mathcal{H}) = (\Sigma\pi) \oplus_x \mathcal{H}$. Note that this is a slight abuse of notation since x is a k-variable in π on the left-hand side, and a $(k+1)$-variable in $\Sigma\pi$ on the right-hand side. Suppose that $f : \Gamma \to B$ is a π-shaped n-term in $\mathbb{V}(\mathcal{C})$. Suppose that Γ is A-simple, and that $x : \mathcal{H}^k A$ is a k-term in Γ for some $k < n$. When $k = n-1$, we define $\mathfrak{J}_x(f)$ by

$$\overline{\mathfrak{J}_x(f)} = \overline{f}, \qquad \mathbf{o}_{\mathfrak{J}_x(f)} = \mathfrak{J}_x(\mathbf{o}_f).$$

Suppose that $k < n-1$, and that $g : s\Gamma \oplus_x \mathcal{H}_A \to sM$ and $h : t\Gamma \oplus_x \mathcal{H}_A \to tM$ are term-wise parallel $(n-1)$-terms in $\mathbb{V}(\mathcal{C})$ such that $\mathfrak{r}_x; g = sf$ and $\mathfrak{r}_x; h = tf$. Then

$$\mathbf{o}_{sf} = \mathbf{o}_{\mathfrak{r}_x;g} = \mathbf{o}_{\mathfrak{r}_x}\,;\mathbf{o}_g, \qquad \mathbf{o}_{tf} = \mathbf{o}_{\mathfrak{r}_x;h} = \mathbf{o}_{\mathfrak{r}_x}\,;\mathbf{o}_h.$$

Hence, we define $\mathfrak{J}_x^{g,h}$ by

$$\overline{\mathfrak{J}_x^{g,h}(f)} = \overline{f}, \qquad \mathbf{o}_{\mathfrak{J}_x^{g,h}(f)} = \mathfrak{J}_x^{\mathbf{o}_g,\mathbf{o}_h}(\mathbf{o}_f).$$

The laws for weak homomorphism types are now satisfied because they are satisfied in $\mathbb{P}$. $\square$

4.3 Homomorphism Types and Two-sided Factorisations

4.3.1 Two-sided Factorisations

Let $\mathcal{C}$ be a category with pullbacks, and let $\mathcal{F}$ be a collection of spans in $\mathcal{C}$. We define $\mathrm{Span}(\mathcal{C}, \mathcal{F})$ to be the subobject of $\mathrm{Span}\,\mathcal{C}$ such that an n-type $M : A \twoheadrightarrow B$ is in $\mathrm{Span}(\mathcal{C}, \mathcal{F})$ when the span M is in $\mathcal{F}$, and whose n-terms are terms in $\mathrm{Span}(\mathcal{C}, \mathcal{F})$ between these spans. We will now describe the relationship between equipping such a globular multicategory with homomorphism types and a notion of *two-sided factorisation.*

Definition 4.3.1.1 ([37]). A *two-sided factorisation* of a span

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A & & B
\end{tikzcd}$$

in $\mathcal{C}$ consists of a factorisation

$$\begin{tikzcd}
& X \arrow[ddl, bend right, "f"'] \arrow[d, "\lambda"] \arrow[ddr, bend left, "g"] & \\
& M \arrow[dl, "\rho_0(f,g)"'] \arrow[dr, "\rho_1(f,g)"] & \\
A & & B
\end{tikzcd}$$

We will follow North [37], and refer to diagrams of this shape as *sprouts*.

Remark 4.3.1.2. Suppose that $f : X \to Y$ is an arrow in the slice category $\mathcal{C}/Z$. Then, to give a factorisation of f in $\mathcal{C}/Z$ is to give a factorisation of f in $\mathcal{C}$. The analogous statement does not hold for two-sided factorisation systems. Suppose that $f, g : X \to A, B$ is a span in $\mathcal{C}/C \times D$. Then to give a factorisation of M in $\mathcal{C}/C \times D$ is to give $\lambda, \rho_0(f,g), \rho_1(f,g)$ making the following diagram commute:

$$\begin{tikzcd}
& X \arrow[ddl, bend right, "f"'] \arrow[d, "\lambda"] \arrow[ddr, bend left, "g"] & \\
& M \arrow[dl, "\rho_0(f,g)"'] \arrow[dr, "\rho_1(f,g)"] & \\
A \arrow[d, "h"'] \arrow[drr, "i", near start] & & B \arrow[d, "j"] \arrow[dll, "k"', near start] \\
C & & D
\end{tikzcd}$$

However, a factorisation of M in $\mathcal{C}$ makes the top triangles commute, but does not guarantee that the bottom rectangles commute. However, when λ is a monomorphism this subtlety disappears; this is the case in many naturally occurring examples where λ is required to be some sort of cofibration.

Remark 4.3.1.3. Suppose that $n \geq 0$ and let $A, B : C \nrightarrow D$ be parallel n-types in $\mathrm{Span}\,\mathcal{C}$. Let $f, g : X \rightrightarrows A, B$ be term-wise parallel n-terms. First suppose that $n = 0$.

Then f, g amounts to a span in $\mathcal{C}$, and to give a two-sided factorisation of this span is to give a 1-term $\lambda : X \to M$, $f \mathrel{\mapstochar\rightarrow} g$,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \lambda"] & X \arrow[d, "g"] \\
A \arrow[r, "|" marking, "M"'] & B
\end{tikzcd}$$

in Span $\mathcal{C}$, filling the diagram above.

Now suppose that $n > 0$. Then, by Example 2.4.0.3, the span f, g corresponds to a span in the slice category $\mathcal{C}/_{C \otimes_{n-1} D}$. Hence, to give a factorisation of this span is to give a 1-term $\lambda : X \to M$, $f \mathrel{\mapstochar\rightarrow} g$ in Span$(\mathcal{C}/_{C \otimes_{n-1} D})$, and this is the same as an $(n+1)$-term $\lambda : X \to M$, $f \mathrel{\mapstochar\rightarrow} g$ in Span $\mathcal{C}$.

Remark 4.3.1.4. Suppose that $\mathcal{C}$ has finite limits. Suppose that $n \geq 0$ and let $A, B : C \mathrel{\mapstochar\rightarrow} D$ be parallel n-types in Span $\mathcal{C}$ as above. Let f, g be a span in the slice category $\mathcal{C}/_{C \otimes_{n-1} D}$. Then to give a factorisation of f, g in $\mathcal{C}/C \otimes_{n-1} D$, it suffices to give a factorisation of f, g in $\mathcal{C}/C \times D$. This follows from the fact that the whole diagram below commutes if and only if the two upper rectangles commute and the two lower rectangles commute.

$$\begin{tikzcd}
 & M \arrow[dl, dashed, "\rho_0(f{,}g)"'] \arrow[dr, dashed, "\rho_1(f{,}g)"] & \\
A \arrow[d, "h"'] \arrow[drr, "i" near start] & & B \arrow[d, "j"] \arrow[dll, "k"' near start] \\
C \arrow[d] \arrow[drr] & & D \arrow[d] \arrow[dll] \\
E & & F
\end{tikzcd}$$

Definition 4.3.1.5. Suppose that we have a sprout

$$\begin{tikzcd}
 & C \arrow[d, "c"] & \\
 & M \arrow[dl, "a"'] \arrow[dr, "b"] & \\
A & & B
\end{tikzcd}$$

together with a span $f, g : X \rightrightarrows Y, B$, and a commutative diagram of solid arrows of the following form:

$$\begin{tikzcd}
C \arrow[r, "x"] \arrow[d, "m"'] & X \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "g"] \\
M \arrow[ur, dashed, "l"'] \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] & \\
A, B \arrow[r, shift left, "y"] \arrow[r, shift right, "z"'] & Y, Z
\end{tikzcd}$$

A *pre-filler* is a dashed arrow l making the bottom square commute. We say that l is a *filler* when it also makes the top triangle commute. We say that a sprout *(pre-)lifts* against a span, when every commutative diagram of this form has a (pre-)filler.

Remark 4.3.1.6. The notion of lifting a sprout against a span is exactly the notion described in [37]. Our notion of pre-filler is a two-sided generalization of the *lower fillers* described in [10, 55].

Remark 4.3.1.7. Translating this definition into a statement about Span $\mathcal{C}$ we find that a 1-term

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow r"] & C \arrow[d, "b"] \\
A \arrow[r, "|" marking, "M"'] & B
\end{tikzcd}$$

pre-lifts against a 1-type $X : Y \nrightarrow Z$ when for any 0-terms $y : A \to Y$ $z : B \to Z$ and 1-term

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "a;y"'] \arrow[dr, phantom, "\Downarrow x"] & C \arrow[d, "b;z"] \\
Y \arrow[r, "|" marking, "X"'] & Z
\end{tikzcd}$$

there is a 1-term $l : M \to X, \quad y \nrightarrow z$.

$$\begin{tikzcd}
A \arrow[r, "|" marking, "M"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow l"] & B \arrow[d, "z"] \\
Y \arrow[r, "|" marking, "X"'] & Z
\end{tikzcd}$$

This pre-lift is a lift when

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow r"] & C \arrow[d, "b"] \\
A \arrow[r, "|" marking, "M"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow l"] & B \arrow[d, "z"] \\
Y \arrow[r, "|" marking, "X"'] & Z
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "a;y"'] \arrow[dr, phantom, "\Downarrow x"] & C \arrow[d, "b;z"] \\
Y \arrow[r, "|" marking, "X"'] & Z
\end{tikzcd}$$

In other words, to give a choice of fillers witnessing that r pre-lifts against X is to give a pre-representation of r relative to X. Furthermore, these pre-lifts are lifts if and only if this pre-representation is a fibrational representation.

Now suppose that $1 < n < m$, and that we have an n-term $r : C \to M$, $a \nrightarrow b$ and an m-type $X : Y \nrightarrow Z$ where $Y, Z : O \nrightarrow P$. Then to give a pre-representation of r against X is to give a pre-lift of the sprout r relative to the span X in $\mathcal{C}/O \otimes_{n-1} P$. Such a term exists when, for any parallel m-terms $y : A \to Y$, $z : B \to Z$, we have a filler

$$\begin{tikzcd}
C \arrow[r, "x"] \arrow[d, "r"'] & X \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "g"] \\
M \arrow[ur, dashed, "l"] \arrow[d, shift right] \arrow[d, shift left] & \\
A, B \arrow[r, shift left, "y"] \arrow[r, shift right, "z"'] \arrow[d, shift right] \arrow[d, shift left] & Y, Z \arrow[dl, Rightarrow] \\
O, P &
\end{tikzcd}$$

in $\mathcal{C}$. However, any pre-filler of the top part of the diagram in $\mathcal{C}$ satisfies the commutativity conditions relating l to O and P automatically, and so defines a pre-filler in $\mathcal{C}/O \otimes_{n-1} P$. Hence, to give a pre-representation of r against X is to give a lift of the sprout r relative to the span X in $\mathcal{C}$. Similarly, to give a fibrational representation of r against X is to give a lift of the sprout r relative to the span X in $\mathcal{C}$.

Definition 4.3.1.8. Consider a diagram of the form

$$\begin{tikzcd}
C \arrow[r, "x"] \arrow[d, "m"'] & X \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "g"] \\
M \arrow[ur, dashed, "l"'] \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] & \\
A, B \arrow[r, shift left, "y"] \arrow[r, shift right, "z"'] & Y, Z
\end{tikzcd}$$

That is a term

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow m"] & C \arrow[d, "b"] \\
A \arrow[r, "M", "|" marking] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow l"] & B \arrow[d, "z"] \\
Y \arrow[r, "X"', "|" marking] & Z
\end{tikzcd}$$

We say that l is a *weak filler*, when $m; l \approx x$ in $\mathrm{Span}(\mathcal{C}, \mathcal{F})$. A *weak lift* structure on a pre-lift is a choice of weak fillers for each pre-filler. It follows that to give a weak lift of a sprout r against a span X in $\mathcal{C}$ is to give a weak representation of r against X in $\mathrm{Span}(\mathcal{C}, \mathcal{F})$.

4.3.2 Homomorphism Type Categories

It is well known that type theories with identity types correspond to categories with classes of maps satisfying suitable factorisation properties. For example, van den Berg and Garner [54] describe a notion of *identity type category* and discuss how these objects can constructed from suitable type theories. Ibid. the authors use these data to describe the weak ω-groupoid structure of the towers of identity types in a type theory. A similar comparison is given in [10, 55], where type theories with path types are compared to *path categories*. We now describe two-sided analogues of these notions, and show how these give rise to globular multicategories with fibrational and weak homomorphism types respectively.

Definition 4.3.2.1. Let $\mathcal{C}$ be a category with finite limits. We say that $\mathcal{C}$ is a *pre-homomorphism type category* when it is equipped with:

- a class $\mathcal{F}$ of spans called *two-sided fibrations*
- a class $\mathcal{R}$ of 1-terms in $\mathrm{Span}_1(\mathcal{C})$ called *representors*.

such that the following conditions hold:

- **Identities** Identity terms are representors.
- **Composition of fibrations:** Whenever $M : A \nrightarrow B$ and $N : B \nrightarrow C$ are two-sided fibrations, their composite $M \otimes_0 N$ is a two-sided fibration.
- **Composition of representors** Whenever $r : M \to M'$ and $s : N \to N'$ are representors such that $r : f \nrightarrow g$ and $s : g \nrightarrow h$, their composite $r \otimes_0 s : M \otimes_0 N \nrightarrow M' \odot N'$ is a representor.
- **Pre-Homomorphism Types:** For each two-sided fibration $M : A \nrightarrow B$, the trivial span $M : M \nrightarrow M$ factorises in $\mathcal{C}/_{A \times B}$ into a sprout $\mathfrak{r}_M : M \to \mathcal{H}_M, \mathrm{id}_M \nrightarrow \mathrm{id}_M$ such that $\mathfrak{r}_M$ is a representor and $\mathcal{H}_M$ is a two-sided fibration.
- **Pre-Lifting:** Whenever $r : M \to N$ is a representor and O is a two-sided fibration, we have a pre-lift lift^r_O of r against O.

Remark 4.3.2.2. The **Composition of representors** property is called the 2-*sided Frobenius condition* in [37].

Proposition 4.3.2.3. *Suppose that* $(\mathcal{C}, \mathcal{F}, \mathcal{R})$ *is a category with pre-homomorphism types. Then the homomorphism types of* $\mathcal{C}$ *correspond to pre-homomorphism types in* $\mathrm{Span}(\mathcal{C}, \mathcal{F})$*.*

Proof. Every substitution $f : \Gamma \to \Delta$ in $\mathrm{Span}(\mathcal{C}, \mathcal{F})$ can be seen as a sprout $\bigotimes f : \bigotimes \Gamma \to \bigotimes \Delta$ in $\mathcal{C}$. We call a term in $\mathrm{Span}(\mathcal{C}, \mathcal{F})$ a representor when this sprout is a representor.

Suppose that $M : A \nrightarrow B$ is an n-type in $\mathrm{Span}(\mathcal{C}, \mathcal{F})$. Then **Pre-Homomorphism Types** induces a two-sided fibration $\mathcal{H}_M$, and this defines an $(n+1)$-type $\mathcal{H}_M : M \nrightarrow M$ in $\mathrm{Span}(\mathcal{C}, \mathcal{F})$. **Pre-Homomorphism Types** also gives us a reflexivity $(n+1)$-term $\mathfrak{r}_M : M \to \mathcal{H}_M$ in $\mathrm{Span}(\mathcal{C}, \mathcal{F})$. This makes $\mathrm{Span}(\mathcal{C}, \mathcal{F})$ reflexive.

Suppose that Γ is an n-context in $\mathrm{Span}(\mathcal{C}, \mathcal{F})$. Suppose that $k < n$, and that x is a k-variable in Γ. Since reflexivity terms are representors and identity sprouts are representors, **Composition of representors** tells us that the substitution $\mathfrak{r}_x^\Gamma$ is a representor. Furthermore, **Composition of fibrations** tells us that $\bigotimes \Gamma$ is a two-sided fibration. Hence, $\bigotimes \mathfrak{r}_x^\Gamma$ pre-lifts against $\bigotimes \Gamma$. By Remark 4.3.1.7 and Remark 4.1.0.3, this induces a choice of pre-homomorphism types for $\mathrm{Span}(\mathcal{C}, \mathcal{F})$. ☐

Definition 4.3.2.4. A *strict homomorphism type category* is a pre-homomorphism type category such that, whenever $r : M \to N$ is a representor and O is a two-sided fibration, the pre-lift lift_O^r of r against O is a lift, and this lift is unique.

Definition 4.3.2.5. A *fibrational homomorphism type category* is a pre-homomorphism type category such that, whenever $r : M \to N$ is a representor and O is a two-sided fibration, the pre-lift lift_O^r of r against O is a lift.

Definition 4.3.2.6. A *weak homomorphism type category* is a pre-homomorphism type category such that, whenever $r : M \to N$ is a representor and O is a two-sided fibration, the pre-lift lift_O^r of r against O can be equipped with the structure of a weak lift.

Theorem 4.3.2.7. *Suppose that* $(\mathcal{C}, \mathcal{F}, \mathcal{I})$ *is a category with pre-homomorphism types.*

1. *When* $(\mathcal{C}, \mathcal{F}, \mathcal{I})$ *is a strict homomorphism type category, the globular multicategory* $\mathrm{Span}(\mathcal{C}, \mathcal{F})$ *has strict homomorphism types.*

2. *When* $(\mathcal{C}, \mathcal{F}, \mathcal{I})$ *is a fibrational homomorphism type category, the globular multicategory* $\mathrm{Span}(\mathcal{C}, \mathcal{F})$ *has fibrational homomorphism types.*

3. *When* $(\mathcal{C}, \mathcal{F}, \mathcal{I})$ *is a weak homomorphism type category, the globular multicategory* $\mathrm{Span}(\mathcal{C}, \mathcal{F})$ *has weak homomorphism types.*

Proof. This follows immediately from Remarks 4.2.0.3, 4.2.0.4 and 4.3.1.7 and Definition 4.3.1.8. □

4.3.3 Construction from Identity Type Categories

We now show that various flavours of categories with homomorphism types are induced by their one-sided analogues.

Definition 4.3.3.1 ([54]). An *identity type category* consists of a category $\mathcal{C}$ together with two classes of morphisms $\mathcal{I}, \mathcal{F} \subseteq \operatorname{Arr} C$, whose elements we refer to as *acyclic cofibrations* and *fibrations* respectively, satisfying the following properties:

- **Fibrancy:** The category $\mathcal{C}$ has a terminal object $\top$, and for each object A, the canonical morphism $A \to \top$ is a fibration.
- **Composition:** The classes $\mathcal{I}$ and $\mathcal{F}$ contain the identities and are closed under composition.
- **Stability:** The pullback of a fibration along an arbitrary morphism in $\mathcal{C}$ exists, and is a fibration.
- **Frobenius:** The pullback of an acyclic cofibration along a fibration is an acyclic cofibration.
- **Orthogonality:** For each acyclic cofibration i, each fibration f, and each commutative square of the form

  $$\begin{tikzcd} C \arrow[r, "x"] \arrow[d, hook, "i"'] & M \arrow[d, two heads, "f"] \\ B \arrow[r, "y"'] \arrow[ur, dashed, "l"] & A \end{tikzcd}$$

  there is a dashed arrow l making the whole diagram commute.
- **Identity Types:** For each fibration $f : M \twoheadrightarrow A$, the diagonal map $\Delta_f : M \to M \times_A M$ factorises into a composite

  $$\begin{tikzcd} M \arrow[r, hook, "\mathfrak{r}_M"] & \mathrm{Id}_M \arrow[r, two heads, "g"] & M \times_A M \end{tikzcd}$$

  where $\mathfrak{r}_M$ is an acyclic cofibration and g is a fibration.

Example 4.3.3.2. The classifying category of any type theory with identity types can be equipped with the structure of an identity type category. (See [54].)

Remark 4.3.3.3. Note that **Fibrancy**, **Stability**, and **Composition** imply that every identity type category has finite products, that product projections are fibrations, and that the product of two fibrations is itself a fibration.

Remark 4.3.3.4. **Stability** implies that every isomorphism is both an acyclic cofibration and a fibration.

Remark 4.3.3.5. By **Fibrancy** and **Stability**, every acyclic cofibration is a split monomorphism.

Theorem 4.3.3.6. *Every identity type category induces a fibrational homomorphism type category such that:*

- *A two-sided fibration $M : A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ is a span such that the corresponding morphism $(a, b) : M \twoheadrightarrow A \times B$ is a fibration.*
- *A representor is a sprout $r : M \to N,\quad f \relbar\joinrel\mapstochar\joinrel\rightarrow g$ whose underlying morphism $r : M \hookrightarrow N$ in $\mathcal{C}$ is an acyclic cofibration.*

Proof. **Identities** follows immediately. **Lifting** implies that we have the pre-lifts required by **Pre-Lifting**, and that these pre-lifts are lifts. Furthermore, **Identity Types** implies **Pre-Homomorphism Types**. Hence, it remains to prove that we can compose two-sided fibrations and representors.

We will first prove two-sided **Composition of Fibrations**, adapting an argument in [47, Proposition 7.2.6]. Suppose that $M : A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ and $N : B \relbar\joinrel\mapstochar\joinrel\rightarrow C$ are two-sided fibrations, corresponding to fibrations $(a, b) : M \twoheadrightarrow A \times B$ and $(b', c) : N \twoheadrightarrow B \times C$ respectively. Then by **Stability**, the left-hand map of the pullback square

$$\begin{tikzcd}
M \times_B N \arrow[r] \arrow[d, two heads, "\mathrm{id}_M \times_B c"'] & N \arrow[d, two heads, "{(b',c)}"] \\
M \times C \arrow[r, "b \times \mathrm{id}_C"'] & B \times C
\end{tikzcd}$$

is a fibration. By **Composition** and Remark 4.3.3.3, we also know that $a \times \mathrm{id}_C$ is a fibration. Hence, by **Composition**, we have that the morphism $a \times_B c : M \times_B N \twoheadrightarrow A \times C$ is a fibration as required.

We next show that we have **Composition of Representors**. Suppose that we have a commutative diagram of spans of the following form:

$$\begin{tikzcd}
 & M \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, hook, "i"] & & N \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, hook, "j"] & \\
A \arrow[dd] & & B \arrow[dd, hook, "k"] & & C \arrow[dd] \\
 & M' \arrow[dl, two heads] \arrow[dr, two heads, "\phi"] & & N' \arrow[dl, two heads, "\psi"'] \arrow[dr, two heads] & \\
A' & & B' & & C'
\end{tikzcd}$$

We need to show that the induced morphism between pullbacks $i \times j : M \times_B N \to M' \times_{B'} N'$ is an acyclic cofibration. We have the following commutative diagram:

$$\begin{tikzcd}
M \times_{B'} N \arrow[r, hook, "\mathrm{id}_M \times j"] \arrow[d, hook, "i \times \mathrm{id}_N"] & M \times_{B'} N' \arrow[r, two heads, "\pi_1"] \arrow[d, hook, "i \times \mathrm{id}_{N'}"] & M \arrow[d, hook, "i"] \\
M' \times_{B'} N \arrow[r, hook, "\mathrm{id}_{M'} \times j"] \arrow[d, two heads, "\pi_2"] & M' \times_{B'} N' \arrow[r, two heads, "\pi_1"] \arrow[d, two heads, "\pi_2"] & M' \arrow[d, two heads, "\phi"] \\
N \arrow[r, hook, "j"] & N' \arrow[r, two heads, "\psi"] & B'
\end{tikzcd}$$

Each quadrant is a pullback square. Since ϕ is a fibration, the downward arrows of the bottom row must be fibrations. Similarly, since ψ is a fibration, the rightward arrows of the second column must be fibrations. Since j is an acyclic cofibration, **Frobenius** implies that is pullback, $\mathrm{id}_{M'} \times j : M' \times_{B'} N \to M' \times_{B'} N$, is also an acyclic cofibration. By a symmetrical argument, the arrow $i \times \mathrm{id}_{N'} : M \times B'N' \to M' \times_{B'} N'$ is also an acyclic cofibration. The composite middle rightward arrow is just the projection $\pi_1 : M' \times_{B'} N' \twoheadrightarrow M'$ and so it is a fibration. Since i is an acyclic cofibration, **Frobenius** implies that its pullback, $i \times \mathrm{id}_N : M \times B'N \to M' \times_{B'} N$ is also an acyclic cofibration. By a symmetrical argument $\mathrm{id}_m \times j : M \times_{B'} N \to M \times_{B'} N'$ is also an acyclic cofibration. **Composition** now implies that the composite $(\mathrm{id}_{M'} \times j) \circ (i \times \mathrm{id}_M) = (i \times \mathrm{id}_{N'}) \circ \mathrm{id}_M \times j : M \times_{B'}$, that is the arrow

$$M \times_{B'} N \xhookrightarrow{i \times j} M' \times_{B'} N',$$

is an acyclic cofibration. Since k is an acyclic cofibration, it is a fortiori a monomorphism. Hence, the canonical morphism $M \times_B N \to M \times_{B'} N$ is an isomorphism. Examining projections, it is clear that the composite

$$M \times_B N \xrightarrow{\cong} M \times_{B'} N \xhookrightarrow{i \times j} M' \times_{B'} N'$$

is precisely the canonical arrow $i \times j : M \times_B N \to M' \times_{B'} N'$. However, by Remark 4.3.3.4 and **Composition**, this composite is an acyclic cofibration. $\square$

Just as categorical models of identity types induce globular multicategories with fibrational homomorphism types, categorical models of path types induce globular multicategories with weak homomorphism types.

Definition 4.3.3.7 ([10,55])**.** Suppose that $\mathcal{C}$ is a category, together with two classes of morphisms $\mathcal{W}, \mathcal{F} \subseteq \operatorname{Arr} C$, whose elements we refer to as *weak equivalences* and *fibrations*. We refer to elements of $\mathcal{W} \cap \mathcal{F}$ as *acyclic fibrations*. We say that $\mathcal{C}$ is a *path category*when it satisfies the following properties:

- **Composition:** Fibrations are closed under composition.
- **Isomorphisms:** Isomorphisms are acyclic fibrations.
- **2-out-of-6:** If $f : A \to B, g : B \to C, h : C \to D$ are composable arrows, and gf and hg are weak equivalences, then so are f, g, h and hgf.
- **Stability:** The pullback of a fibration along an arbitrary morphism in $\mathcal{C}$ exists, and is again a fibration. The pullback of an acyclic fibration along an arbitrary morphism in $\mathcal{C}$ exists, and is again an acyclic fibration.
- **Path objects:** For each object $A \in \mathcal{C}$, the diagonal map $\Delta_A : A \to A \times A$ factorises into a composite

  $$\begin{tikzcd} A \arrow[r, hook, "\mathfrak{r}_M"] & \mathrm{Id}_A \arrow[r, two heads, "(s,t)"] & A \times A \end{tikzcd}$$

  where $\mathfrak{r}_M$ is an acyclic cofibration and g is a fibration.
- **Fibrancy:** The category $\mathcal{C}$ has a terminal object $\top$, and for each object A, the canonical morphism $A \to \top$ is a fibration.
- **Cofibrancy:** Every acyclic fibration has a section.

Example 4.3.3.8. The classifying category of any type theory with propositional identity types can be equipped with the structure of a path category. See [10].

Example 4.3.3.9. Let $\mathcal{C}$ be a Quillen model category. If every object of $\mathcal{C}$ is cofibrant, then the subcategory of fibrant objects in $\mathcal{C}$ is a path-category. In particular, both the Kan-Quillen model structure, and the Joyal model structure on simplicial sets satisfy this property. Fibrant objects in these cases are Kan complexes and quasi-categories respectively. Hence, standard topological models of $(\infty, 0)$- and $(\infty, 1)$-categories can be organized into path categories.

Theorem 4.3.3.10. *Every path category induces a weak homomorphism type category such that:*

- *A two-sided fibration* $M : A \mathrel{\mkern3mu\mapstochar\mkern-3mu\rightarrow} B$ *is a span such that the corresponding morphism* $(a, b) : M \twoheadrightarrow A \times B$ *is a fibration.*
- *A representor is a sprout* $r : M \to N, f \mathrel{\mkern3mu\mapstochar\mkern-3mu\rightarrow} g$ *whose underlying morphism* $r : M \hookrightarrow N$ *in* $\mathcal{C}$ *is a weak equivalence.*

Proof. **Composition of Fibrations** follows by the same argument given in the proof of Theorem 4.3.3.6. **Pre-Homomorphism Types** follows from [55][Proposition 2.3]. **Pre-lifting** follows from [55][Lemma 2.9], and these pre-lifts can be made into weak lifts using [55][Theorem 2.38].

Finally, **Composition of Representors** amounts to the following well known result: if we have a transformation between cospans whose objects are fibrant, and whose legs are fibrations

$$\begin{tikzcd}
M \arrow[dr, two heads] \arrow[dd, "\sim"', "i"] & & N \arrow[dl, two heads] \arrow[dd, "\sim"', "j"] \\
& B \arrow[dd, "\sim"', "k"] & \\
M' \arrow[dr, two heads, "\phi"'] & & N' \arrow[dl, two heads, "\psi"] \\
& B' &
\end{tikzcd}$$

such that the vertical maps are weak equivalences, then the induced map between pullbacks $i \times j : M \times_B N \to M' \times_{B'} N'$ is a weak equivalence. (These pullbacks are homotopy pullbacks.) □

Chapter 5

Constructing Higher Categories

We have seen a number of manifestations of the close relationship between globular multicategories with homomorphism types and higher categories. Various collections of higher categories give rise to globular multicategories with homomorphism types. Furthermore, each higher category induces a globular multicategory of elements, and, under mild conditions, a vertical globular multicategory. We now aim to demonstrate results in the opposite direction: given globular multicategories with homomorphism types, we will construct higher categorical structures.

To this end, we study the structures attached to each type and term in globular multicategories with homomorphism types. For each n, there is a globular multicategory $L_{\mathcal{H}}\mathbb{T}_n$ with strict homomorphism types such that to give an n-type in a globular multicategory $\mathbb{X}$ with strict homomorphism types is to give a homomorphism type preserving homomorphism

$$L_{\mathcal{H}}\mathbb{T}_n \longrightarrow \mathbb{X}.$$

Viewing globular multicategories as algebraic theories this correspondence says that each n-term in $\mathbb{X}$ is a model of $L_{\mathcal{H}}\mathbb{I}^n_\pi$, or, equivalently, that the theory $L_{\mathcal{H}}\mathbb{I}^n_\pi$ is the *theory of n-types with strict homomorphism types.* Similarly, for each n-pasting diagram π, there is a globular multicategory $L_{\mathcal{H}}\mathbb{I}^n_\pi$ with strict homomorphism types such that to give a π-shaped n-term in a globular multicategory $\mathbb{X}$ with strict homomorphism types is to give a homomorphism type preserving homomorphism

$$L_{\mathcal{H}}\mathbb{I}^n_\pi \longrightarrow \mathbb{X}.$$

Hence, the globular multicategory $L_{\mathcal{H}}\mathbb{I}^n_\pi$ can be seen as the *theory of n-terms with strict homomorphism types.* Thus, types and terms inside a globular multicategory with strict homomorphism types can be understood by studying $L_{\mathcal{H}}\mathbb{T}_n$ and $L_{\mathcal{H}}\mathbb{I}^n_\pi$.

We show that $L_{\mathcal{H}}\mathbb{I}_0 = \mathbb{1}$, the terminal globular operad. Since $\mathbb{1}$ is well known to be, in a precise sense, the *theory of strict ω-categories*, it follows that each every 0-type in a globular multicategory $\mathbb{X}$ with strict homomorphism types has the structure of a strict ω-category in $\mathbb{X}$. We show that, moreover, when $\mathbb{X} = \operatorname{Mod}\mathbb{Y}$ for some $\mathbb{Y}$, the 0-types in $\mathbb{Y}$ are exactly strict ω-categories in $\mathbb{X}$. See Example 5.2.0.2.

In a similar vein, we show that $L_{\mathcal{H}}\mathbb{I}_0^0$ can be seen as the *theory of strict functors between strict ω-categories*. Thus, every 0-term $f : A \to B$ in a globular multicategory $\mathbb{X}$ with strict homomorphism types is, in a precise sense, a functor between the strict ω-categories in $\mathbb{X}$ corresponding to A and B. See Example 5.2.0.4.

These results and more serve to demonstrate a deep connection between strict homomorphism types and strict higher categorical structures. Our next goal is to describe a similar relationship between fibrational homomorphism types and certain weak higher categorical structures.

It is well known that contractible globular operads can be seen as *theories of weak ω-categories* (see [6]). However, other weak higher categorical structures such as higher functors are less well understood in the globular setting. See for instance [23, 25, 26] for work in this direction. We develop a new approach to understanding these objects. We define *acyclic fibrations of globular multicategories*; whenever $\mathbb{Y} \twoheadrightarrow \mathbb{X}$ is an acyclic fibration and $\mathbb{X}$ is the "theory of widgets" we view $\mathbb{Y}$ as a "theory of weak widgets". In particular, a globular operad $\mathbb{P}$ is contractible exactly when the canonical homomorphism $\mathbb{P} \to \mathbb{1}$ is an acyclic fibration. This last result follows from [22].

Having developed tools to understand weak higher categorical structures in globular multicategories, we then relate these structures to fibrational homomorphism types. We mirror the approach taken for strict homomorphism types. For each n, there is a globular multicategory $L_{\mathcal{H}}^{\mathrm{Fib}}\mathbb{T}_n$ with fibrational homomorphism types such that to give an n-type in a globular multicategory $\mathbb{X}$ with fibrational homomorphism types is to give a homomorphism type preserving homomorphism

$$L_{\mathcal{H}}^{\mathrm{Fib}}\mathbb{T}_n \longrightarrow \mathbb{X}.$$

Hence, $L_{\mathcal{H}}^{\mathrm{Fib}}\mathbb{I}_\pi^n$ is the *theory of n-types with fibrational homomorphism types*. Similarly, for each n-pasting diagram π, there is a globular multicategory $L_{\mathcal{H}}^{\mathrm{Fib}}\mathbb{I}_\pi^n$ with strict homomorphism types such that to give a π-shaped n-term in a globular multicategory $\mathbb{X}$ with strict homomorphism types is to give a homomorphism type preserving homomorphism

$$L_{\mathcal{H}}^{\mathrm{Fib}}\mathbb{I}_\pi^n \longrightarrow \mathbb{X}.$$

Hence, the globular multicategory $L^{\mathrm{Fib}}_{\mathcal{H}}\mathbb{I}^n_\pi$ can be seen as the *theory of n-terms with fibrational homomorphism types*. We show that there are canonical acyclic fibrations $L^{\mathrm{Fib}}_{\mathcal{H}}\mathbb{I}_n \to L_{\mathcal{H}}\mathbb{I}_n$ and $L^{\mathrm{Fib}}_{\mathcal{H}}\mathbb{I}^n_\pi \to L_{\mathcal{H}}\mathbb{I}^n_\pi$. Thus, the theory of n-types with fibrational homomorphism types is a weakening of the theory of n-types with strict homomorphism types, and similar statements hold for theories of terms with homomorphism types. In particular, every 0-type in a globular multicategory with fibrational homomorphism types has the structure of a weak ω-category, and every 0-term can be seen as a weak ω-functor between these categories. See Example 5.4.0.5 and Example 5.4.0.6. We make a conjecture that would allow us to obtain similar results about globular multicategories with weak homomorphism types.

5.1 Shapes of Types and Terms

Every globular multigraph can be viewed as a presheaf over a category whose objects are shapes of types and terms. We can better understand the structure of types and terms in globular multicategories with homomorphism types by describing how to freely add homomorphism types to the representables induced by these objects.

Definition 5.1.0.1. We define a category $\mathbb{G}^+$ of *generic types and terms*. Its set of objects is the coproduct of sets

$$\mathbb{G} + \mathrm{el}(\mathbf{pd}).$$

Thus, for each $n \in \mathbb{G}$, there is an object $\mathbb{T}_n$ in $\mathbb{G}^+$, and for each $\pi \in \mathbf{pd}(n)$, there is an object $\mathbb{I}^n_\pi$ in $\mathbb{G}^+$. We refer to $\mathbb{T}_n$ as the *generic n-type*, and we refer to $\mathbb{I}^n_\pi$ as the *generic π-shaped n-term*. There are four classes of arrows in $\mathbb{G}^+$:

- Every arrow $\sigma : m \to n$ in $\mathbb{G}$ induces a corresponding arrow $\mathbb{T}_m \to \mathbb{T}_n$ between generic types in $\mathbb{G}^+$. These arrows pick out the source and target types of generic types.

- Every arrow $\sigma : m \to n$ in $\mathrm{el}(\mathbf{pd})$ induces a corresponding arrow $\mathbb{I}^\sigma_\pi : \mathbb{I}^m_{\sigma\pi} \to \mathbb{I}^n_\pi$ between generic terms in $\mathbb{G}^+$. These arrows pick out the source and target terms of generic terms.

- For each $k \leq n$, each n-pasting diagram π, each map of globular sets $x : k \to \pi$ induces an arrow

  $$\mathbf{V}_x : \mathbb{T}_k \longrightarrow \mathbb{I}^n_\pi$$

  in $\mathbb{G}^+$. These arrow pick out the variables (types) in the domain contexts of generic terms.

- For each $k \leq n$, each n-pasting diagram π, and π,and each arrow $A : k \to n$ in $\mathbb{G}$, there is an arrow
$$\mathbf{O}^A_\pi : \mathbb{T}_k \longrightarrow \mathbb{I}^n_\pi$$
in $\mathbb{G}^+$.These arrows pick out the output types of generic terms.

Composition of arrows in $\mathbb{G}^+$ is induced by composition in $\mathbb{G}$-Set.

Theorem 5.1.0.2. *The category of globular multigraphs is equivalent to the category of presheaves over* $\mathbb{G}^+$*.*

Proof. This follows from [30, Proposition C.3.4 and Proposition 6.5.6]. □

Remark 5.1.0.3. Each object of $\mathbb{G}^+$ can be viewed as a globular multigraph or as a globular multicategory. The globular multigraph corresponding to $\mathbb{T}_n$ has a unique non-degenerate n-type. The globular multigraph corresponding to $\mathbb{I}^n_\pi$ has a unique n-term h_π. This n-term is π-shaped. Each type A in $\mathbb{I}^n_\pi$ corresponds to either:

- a cell in the pasting diagram π if A is in the source context of h_π,
- or a cell in the representable n if A is in the target type of h_π.

Suppose that $\mathbb{X}$ is a globular multicategory. Then, by the Yoneda Lemma, an n-type A in $\mathbb{X}$ corresponds to a homomorphism $\bar{A} : \mathbb{T}_n \to \mathbb{X}$, and a π-shaped an n-term f in $\mathbb{X}$ corresponds to a homomorphism $\bar{f} : \mathbb{I}^n_\pi \to \mathbb{X}$.

Remark 5.1.0.4. The composition of terms in $\mathbb{X}$ can also be described using operations on the generic types and terms. Let f, g be a pair of composable 0-terms in $\mathbb{X}$. Consider the following pushout of generic terms:

$$\begin{tikzcd}
\mathbb{T}_0 \arrow[r, "\mathbf{O}^\star_0"] \arrow[d, "\mathbf{V}_\star"'] & \mathbb{I}^0_0 \arrow[d, "\iota_1"] \\
\mathbb{I}^0_0 \arrow[r, "\iota_2"'] & \mathbb{I}^0_0 +_0 \mathbb{I}^0_0
\end{tikzcd}$$

Then the pair $\bar{f}, \bar{g}$ induces a canonical homomorphism $(\bar{f}, \bar{g}) : \mathbb{I}^0_0 +_0 \mathbb{I}^0_0 \to \mathbb{X}$ such that $(\bar{f}, \bar{g}) \circ \iota_1 = \bar{f}$ and $(\bar{f}, \bar{g}) \circ \iota_2 = \bar{g}$. There is a canonical homomorphism

$$\mathbb{I}^0_0 \xrightarrow{c} \mathbb{I}^0_0 +_0 \mathbb{I}^0_0$$

that "picks out the composite". Let $h : A \to B$ be the unique non-trivial term in $\mathbb{I}^0_0$. Then $c(h) = \iota_1(h); \iota_2(h)$. It follows that $(\bar{f}, \bar{g}) \circ c = \overline{f; g}$.

More generally, let π be an n-pasting diagram. Then π induces a diagram $\Pi : \mathrm{el}(\pi) \to$ GlobMult such that

- For each $i \in \pi(k)$, we have that $\Pi(i) = \mathbb{T}_k$.
- For each $\sigma : (k,i) \to (k',j)$ in $\mathrm{el}(\pi)$, we have that $\Pi(\sigma) = \mathbb{T}_\sigma : \mathbb{T}_i \to \mathbb{T}_j$.

Let $\mathbb{T}_\pi$ be the colimit of Π. By construction, there is a canonical homomorphism $\mathbf{V}_\pi : \mathbb{T}_\pi \to \mathbb{I}^n_\pi$ such that the restriction of $\mathbf{V}_\pi$ to the component of $\mathbb{T}_\pi$ corresponding to $i \in \pi(k)$ is $\mathbf{V}_i$. Intuitively, this arrow picks out the source context of the generic π-shaped n-term. Suppose that we have a π-shaped pasting diagram of pasting diagrams $(\rho_i)_{i\in\pi}$. Let $P : \mathrm{el}(\pi) \to$ GlobMult be defined so that:

- For each $i \in \pi(k)$, we have that $P(i) = \mathbb{I}^k_{\rho_i}$.
- For each $\sigma : (k,i) \to (k',j)$ in $\mathrm{el}(\pi)$, we have that $P(\sigma) = \mathbb{I}^\sigma_{\rho_i} : \mathbb{I}^k_{\rho_i} \to \mathbb{I}^{k'}_{\rho_j}$.

Let $\mathbb{I}^\pi_\rho$ be the colimit of P. For each $i \in \pi(k)$, there is a homomorphism $\mathbf{O}^k_{\rho_i} : \mathbb{T}_k \to \mathbb{I}^k_{\rho_i}$. These together induce a canonical homomorphism $\mathbf{O}^\pi_\rho : \mathbb{T}_\pi \to \mathbb{I}^\pi_\rho$.

Hence, consider the following pushout in GlobMult:

$$\begin{tikzcd}
\mathbb{T}_\pi \arrow[r, "\mathbf{O}^\pi_\rho"] \arrow[d, "\mathbf{V}_\pi"'] & \mathbb{I}^\pi_\rho \arrow[d, "\iota_1"] \\
\mathbb{I}^n_\pi \arrow[r, "\iota_2"'] & \mathbb{I}^\pi_\rho +_\pi \mathbb{I}^n_\pi
\end{tikzcd}$$

Let Δ be a π-shaped n-context in $\mathbb{X}$. Then Δ corresponds to a homomorphism $\overline{\Delta} : \mathbb{T}_\pi \to \mathbb{X}$. Let $f : \Gamma \to \Delta$, $g : \Delta \to A$ be a pair of composable n-terms in $\mathbb{X}$ such that f_i is ρ_i-shaped. Then f corresponds to a homomorphism $\bar{f} : \mathbb{I}^\pi_\rho \to \mathbb{X}$, and g corresponds to a homomorphism $\bar{g} : \mathbb{I}^n_\pi \to \mathbb{X}$. It follows that there is an induced homomorphism $(\bar{f}, \bar{g}) : \mathbb{I}^\pi_\rho +_\pi \mathbb{I}^n_\pi \to \mathbb{X}$. Let h_π be the unique non-trivial n-term in $\mathbb{I}^n_\pi$. Then $\bar{f}(h_\pi) = f$. For each $i \in \pi(k)$, let h_{ρ_i} be the unique non-trivial k-term in $\mathbb{I}^k_{\rho_i}$. Then $\bar{g}(h_{\rho_i}) = g_i$. Furthermore, there is a canonical composite $h_\rho; h_\pi$ in the pushout $\mathbb{I}^\pi_\rho +_\pi \mathbb{I}^n_\pi$. Let $\sigma = \bigodot_{i\in\pi} \rho_i$. Let

$$c = \overline{h_\rho; h_\pi} : \mathbb{I}^n_\sigma \longrightarrow \mathbb{I}^\pi_\rho +_\pi \mathbb{I}^n_\pi.$$

Then $(\bar{f}, \bar{g}) \circ c = \overline{f; g}$.

Remark 5.1.0.5. The category $\mathbb{G}^+$ is a *direct category.* Let $\mathbb{N}$ be the poset of natural numbers. There is an identity-reflecting functor $\dim : \mathbb{G}^+ \to \mathbb{N}$ which sends the generic n-type and all generic n-terms to the natural number n. Let $\mathbb{U} \in \mathbb{G}^+$ be a generic n-type or n-term, identified with its image under the Yoneda embedding. Then the *boundary* $\partial\mathbb{U}$ is the subpresheaf of $\mathbb{U}$ such that

$$w \in \partial\mathbb{U}(v) \iff \dim v < n.$$

We will denote the canonical inclusion of the boundary by

$$\iota_{\mathbb{U}} : \partial\mathbb{U} \longrightarrow \mathbb{U}.$$

In Section 5.3 we will use these boundary inclusions to describe a higher dimensional notion of weakness.

Definition 5.1.0.6. Let $\overline{\mathrm{GlobGraph}}$ be the category of globular multigraphs with a reflexive globular set of types. Suppose that $\mathbb{X}$ is an object in $\overline{\mathrm{GlobGraph}}$. Suppose that $0 \leq k \leq n$, and let π be an n-shaped pasting diagram. Suppose that Γ is a π-shaped n-context, that M is an n-type, and that $g : s\Gamma \to sM\, h : t\Gamma \to tM$ are term-wise parallel $(n-1)$-terms. Then we refer to a k-cell in $[[\Gamma \to M, g \nrightarrow h]]$ as a *π-shaped k-transfor*. We define the *generic π-shaped k-transfor* $\mathbb{I}^{\pi}_{\mathcal{H}^k n}$ to be the initial globular multigraph with a reflexive globular set of types containing a π-shaped k-transfor. When $k = 0$, this is just the generic term $\mathbb{I}^{\pi}_{n}$. When $k > 0$, the globular multigraph $\mathbb{I}^{\pi}_{\mathcal{H}^k_n}$ can be constructed by quotienting $\mathbb{I}^{\pi}_{n+k}$ so that its unique $(n+k)$-type is $\mathcal{H}^k M : \mathcal{H}^{k-1}M \nrightarrow \mathcal{H}^{k-1}M$. Then to give a homomorphism,

$$\mathbb{I}_{\pi}^{\mathcal{H}^k_n} \longrightarrow \mathbb{X}$$

preserving the reflexive structure on types, is to give a π-shaped k-transfor in $\mathbb{X}$. We define the boundary $\partial\mathbb{I}_{\pi}^{\mathcal{H}^k_n}$ just as for generic terms.

5.2 Strict Higher Categories from Strict Homomorphism Types

We now study results justifying the intuition that objects in globular multicategories with homomorphism types are "higher category-like". Given globular multicategories with strict homomorphism types, we construct strict higher categorical structures.

Let $\mathbb{X}$ and $\mathbb{Y}$ be globular multicategories, and suppose that $\mathbb{X}$ has strict homomorphism types. Let $U_{\mathcal{H}}\,\mathrm{GlobMult}_{\mathcal{H}} \to \mathrm{GlobMult}$ be the functor forgetting homomorphism types. Then, since $U_{\mathcal{H}}$ has a left adjoint, $L_{\mathcal{H}}$, we have a natural bijection between homomorphisms

$$\mathbb{Y} \longrightarrow U_{\mathcal{H}}\mathbb{X}$$

and homomorphism type preserving homomorphisms

$$L_{\mathcal{H}}\mathbb{Y} \longrightarrow \mathbb{X}.$$

When $\mathbb{X} = \operatorname{Mod}\mathbb{Z}$ is the result of the modules construction, applying adjointness again, such a homomorphism corresponds to a homomorphism

$$U_{\mathcal{H}} L_{\mathcal{H}} \mathbb{Y} \longrightarrow \mathbb{Z}.$$

In particular, when $\mathbb{Y} = \mathbb{T}_n$ is a generic type, a homomorphism $\mathbb{T}_n \to U_{\mathcal{H}}\mathbb{X}$ is just a type in $\mathbb{X}$, and so we can understand types in $\mathbb{X}$ by describing $L_{\mathcal{H}}\mathbb{T}_n$. Similarly, we can understand terms in $\mathbb{X}$ by describing $L_{\mathcal{H}}\mathbb{I}^n_\pi$.

Remark 5.2.0.1. Let $\mathbb{X}$ be a globular multicategory. Then the types and terms of $L_{\mathcal{H}}\mathbb{X}$ are inductively generated by the following rules:

- For each type (or term) A in $\mathbb{X}$, there is a canonical type (or term) A in $L_{\mathcal{H}}\mathbb{X}$.
- Whenever A is an n-type in $L_{\mathcal{H}}\mathbb{X}$, there is a canonical $(n+1)$-type $\mathcal{H}_A$ and a $(n+1)$-term $\mathfrak{r}_A : A \to \mathcal{H}_A, \mathrm{id}_A \nrightarrow \mathrm{id}_A$ satisfying reflexivity rules.

Hence, each type in $L_{\mathcal{H}}\mathbb{X}$ is of the form $\mathcal{H}^i_A$ for some $A \in \mathbb{X}$ and $i \geq 0$. Let $f : \Gamma \to \mathcal{H}^i_A$ be a term in $L_{\mathcal{H}}\mathbb{X}$. Then Γ must be of the form $\Gamma' \oplus_{x_1} \mathcal{H}_{B_1} \oplus_{x_2} \cdots \oplus_{x_l} \mathcal{H}_{B_l}$, for some sequence of variables $x_1 : B_1, \ldots, x_l : B_l$. Precomposing with reflexivity terms, we may obtain a term

$$\mathfrak{n}(f) : \Gamma' \to \mathcal{H}^i A,$$

and by induction we must have that

$$\mathfrak{n}(f) = \mathfrak{g}(f); \mathfrak{r}^i_A,$$

for some $\mathfrak{g}(f) \in \mathbb{X}$. Thus f is of the form

$$\mathfrak{J}_{x_1,\ldots,x_l}(\mathfrak{g}(f); \mathfrak{r}^i_A),$$

for some variables $x_1, \ldots, x_l$, and $i \geq 0$. In fact, every term of $L_{\mathcal{H}}f$ is uniquely determined by this data.

Example 5.2.0.2. The globular multicategory $\mathbb{T}_0$ has a unique 0-type $\star$ and a unique 0-term, $\mathrm{id}_\star$. Thus, $L_{\mathcal{H}}\mathbb{T}_0$ contains a unique n-type, $\mathcal{H}^n_\star$, for each n. Suppose that $f, f' : \Gamma \to \mathcal{H}^i_\star$ are parallel n-terms in $L_{\mathcal{H}}\mathbb{T}_0$. Then we must have that $\mathfrak{g}(f) = \mathfrak{g}(f') = \mathrm{id}_\star$. Since f, f' are parallel, it now follows that $f = f'$. On the other hand, for any $n \geq 0$ and any n-dimensional pasting diagram π, we can use $\mathfrak{J}$-terms to construct π-shaped n-term $\mathfrak{J}_\pi(\mathfrak{r}^n_\star)$ in $L_{\mathcal{H}}\mathbb{T}_0$. Hence, $L_{\mathcal{H}}\mathbb{T}_0$ is the terminal globular operad $\mathbb{1}$. Thus, every 0-type of a globular multicategory with strict homomorphism types has the structure of a strict ω-category.

Example 5.2.0.3. When $n > 0$, we think of $L_{\mathcal{H}}\mathbb{T}_n$ as a theory of higher profunctors. It follows from the description of $\mathrm{Mod}\,\mathrm{SpanSet}$ in Example 3.5.3.4 that an algebra of $\mathbb{T}_1$ is precisely a profunctor enriched in $\mathrm{Str}\,\omega$-Cat. Thus, every 1-type in a globular multicategory with strict homomorphism types has the structure of a strict ω-profunctor.

Example 5.2.0.4. The globular multicategory $\mathbb{I}_0^0$ contains exactly two 0-types A and B and its only non-trivial 0-term is a term $f_0 : A \to B$. Thus, the types of $L_{\mathcal{H}}\mathbb{I}_0^0$ are of the form $\mathcal{H}_A^k, \mathcal{H}_B^k$ for each k. The terms of $L_{\mathcal{H}}Str\mathbb{I}_0^0$ can be divided into three classes:

- The collection of terms such that $\mathfrak{g}(f) = \mathrm{id}_A$ assemble into a copy of the terminal globular operad $\mathbb{1}$.
- The collection of terms such that $\mathfrak{g}(f) = \mathrm{id}_B$ assemble into another copy of the terminal globular operad $\mathbb{1}$.
- Suppose that f is a term in $L_{\mathcal{H}}\mathbb{I}_0^0$ such that $\mathfrak{g}(f) = f_0$. Let π be a k-pasting diagram. Let π_A be the unique π-shaped context in the terminal globular operad generated by A. Then there is a unique term $\pi_A \to \mathcal{H}_B^k$ namely $\mathfrak{J}_\pi(f_0; \mathfrak{r}_B^k)$.

Let $F : U_{\mathcal{H}}L_{\mathcal{H}}\mathbb{I}_0^0 \to \mathrm{Span}(\mathrm{Set})$ be an algebra of $U_{\mathcal{H}}L_{\mathcal{H}}\mathbb{I}_0^0$. Then these collections induce:

- An ω-category $F(A)$
- An ω-category $F(B)$
- For each k-pasting diagram π_A in $F(A)$, we have a unique assignment sending π_A to the k-type $\mathcal{H}^k B$. In other words, we have a strict ω-functor $F(A) \to F(B)$.

Thus, we think of $L_{\mathcal{H}}\mathbb{I}_0^0$ as the *theory of strict ω-functors*. It follows that every 0-term in a globular multicategory with strict homomorphism types has the structure of a strict ω-functor.

Example 5.2.0.5. Let $L_{\mathcal{H}}\mathbb{I}_{\mathcal{H}_0}^0$ be the generic k-transfor between 0-terms with strict homomorphism types. It follows from Example 3.5.3.4 that an algebra of $L_{\mathcal{H}}\mathbb{I}_0^{\mathcal{H}_0}$ is a strict natural transformation between strict ω-functors.

Remark 5.2.0.6. By taking truncations, we obtain similar descriptions of the n-globular multicategories whose algebras are strict n-categories, strict profunctors, between these categories, as well as strict higher n-functors and strict higher transformations.

Suppose that $\mathbb{X}$ is a globular multicategory with strict homomorphism types. Suppose that B is a 0-context in $\mathbb{X}$, and that A is a 0-type in $\mathbb{X}$. Then the strict ω-category structure on A induces a strict ω-category structure on globular set $[[\Gamma \to A]]$ of 0-terms with codomain A. We define a strict ω-category $\mathcal{C} : \mathbb{1} \to \mathrm{SpanSet}$ as follows:

- For each n, we set $\mathcal{C}(n) = [[B \to A]](n)$.
- Let $A' : \mathbb{1} = L_{\mathcal{H}}0 \to \mathbb{X}$ be the homomorphism corresponding to A. Let π be an n-pasting diagram, and let $\mathbf{c}_\pi : \pi \to n$ be the unique π-shaped term in $\mathbb{1}$. For each n-pasting diagram π, a map $p : \pi \to [[\Gamma \to A]]$ is equivalently a substitution $p : \Gamma \to A'(\pi)$ in $\mathbb{X}$. Hence, we define the operation $\mathcal{C}(\mathbf{c}_\pi)$ by

$$\mathcal{C}(\mathbf{c}_\pi)(p) = p; A'(\mathbf{c}_\pi).$$

Now suppose that A, B are parallel n-types in $\mathbb{X}$. Then $\mathbb{X}(A, B)$ is a globular multicategory with strict homomorphism types. Each n-type $M : A \twoheadrightarrow B$ in $\mathbb{X}$ is a 0-type in $\mathbb{X}(A, B)$, and consequently we have a homomorphism $M' : \mathbb{1} = L_{\mathcal{H}}0 \to \mathbb{X}(A, B)$. Arguing as above, the homomorphism M' allows us to equip the globular set

$$[[\Gamma \longrightarrow M, \quad g \twoheadrightarrow h]]$$

with the structure of a strict ω-category for each n-context Γ, and each pair of termwise parallel $(n-1)$-types $g : s\Gamma \to A$ and $h : t\Gamma \to B$.

5.3 Homotopical Tools for Globular Multicategories

By Remark 5.1.0.5, the category $\mathbb{G}^+$ of generic types and terms is a direct category. This induces a weak factorisation system on globular multicategories and related structures. We will use this weak factorisation system to understand the structure of objects in globular multicategories with fsbrational homomorphism types.

Definition 5.3.0.1. Let us denote the set of boundary inclusions of $\mathbb{G}^+$ by

$$I = \{\iota_{\mathbb{U}} : \partial\mathbb{U} \longrightarrow \mathbb{U} \mid \mathbb{U} \in \mathbb{G}^+\}.$$

Then I cofibrantly generates a weak factorisation system $(\mathcal{L}, \mathcal{R})$ on the category of globular multigraphs GlobGraph. We refer to maps in $\mathcal{L}$ as *cofibrations* and maps in $\mathcal{R}$ as *acyclic fibrations*. A map of globular multigraphs $f : X \to \mathbb{Y}$ is an acyclic fibration when, for any generic type or term $\mathbb{U}$, each commutative square

$$\begin{tikzcd}
\partial\mathbb{U} \arrow[r] \arrow[d, hook, "\iota_{\mathbb{U}}"'] & \mathbb{X} \arrow[d, two heads, "\mathbb{F}"] \\
\mathbb{U} \arrow[r] \arrow[ru, dotted] & \mathbb{Y}
\end{tikzcd}$$

has a filler. A map of globular multigraphs $i : Z \to W$ is a cofibration when for each acyclic fibration $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$, each commutative square

$$\begin{tikzcd}
\partial\mathbb{Z} \arrow[r] \arrow[d, hook, "i"'] & \mathbb{X} \arrow[d, two heads, "\mathbb{F}"] \\
\mathbb{W} \arrow[r] \arrow[ru, dotted] & \mathbb{Y}
\end{tikzcd}$$

has a filler.

Proposition 5.3.0.2. *A map of globular multigraphs is a cofibration exactly when it is a monomorphism.*

Proof. Since $\mathbb{G}^+$ is a direct category, it is skeletal and has no non-trivial automorphisms. The result now follows from [15, Proposition 8.1.37]. □

Remark 5.3.0.3. A similar argument works for maps of globular multigraphs with a reflexive globular set of types. Suppose that $\mathbb{I}_\pi^{\mathcal{H}_n^k}$ is the generic k-transfor between π-shaped k-terms. Then it follows that the boundary inclusion of $\mathbb{I}_\pi^{\mathcal{H}_n^k}$ is a cofibration.

This weak factorisation system can be transferred to other categories of interest using the adjunctions induced by various forgetful functors. We have the following commutative diagram of forgetful functors:

$$\begin{tikzcd}
 & \mathrm{GlobMult}_{\mathcal{H}} \arrow[d] & \\
 & \mathrm{GlobMult}_{\mathcal{H}}^{\mathrm{Fib}} \arrow[d] & \\
 & \mathrm{GlobMult}_{\mathcal{H}}^{\mathrm{Pre}} \arrow[ld] \arrow[rd] & \\
\mathrm{GlobMult} \arrow[rd] & & \overline{\mathrm{GlobGraph}} \arrow[ld] \\
 & \mathrm{GlobGraph} &
\end{tikzcd}$$

Each of these forgets essentially algebraic data and so has a left adjoint. Let $U : \mathcal{C} \to \mathcal{D}$ be one of these forgetful functors, and let $F : \mathcal{D} \to \mathcal{C}$ be its left adjoint. Then the weak factorisation system of *cofibrations* and *acyclic fibrations* in $\mathcal{C}$ is generated by

$$F\iota_{\mathbb{U}} : F\partial\mathbb{U} \hookrightarrow F\mathbb{U}$$

for each generating cofibration in ι_u in $\mathcal{C}$. A morphism $f : \mathbb{X} \to \mathbb{Y}$ in $\mathcal{C}$ is an acyclic fibration exactly when Uf is an acyclic fibration in $\mathcal{D}$. Moreover, the left adjoint F preserves cofibrations.

Example 5.3.0.4. Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are globular operads. Then every homomorphism of globular operads is bijective on types and so the lifting conditions for generic types are always satisfied. It follows that a homomorphism $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ is an acyclic fibration if and only if it satisfies the lifting conditions for generic terms. The canonical map

$$\mathbb{X} \xrightarrow{!} \mathbb{1}$$

to the terminal operad is an acyclic fibration exactly when $\mathbb{X}$ is a normalised contractible globular operad. This follows from the observations of Garner in [22].

We now give a useful alternative description of acyclic fibrations. Intuitively, this description says that the term-lifting properties of acyclic fibrations are satisfied exactly when, on terms, a homomorphism is strictly surjective and weakly reflects identities.

Definition 5.3.0.5. Let $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ be a homomorphism of globular multicategories with pre-homomorphism types. We say that $\mathbb{F}$ *weakly reflects identities of terms* if, for all parallel terms $v, v' : \Gamma \to A$ in $\mathbb{X}$ such that $\mathbb{F}(v) = \mathbb{F}(v')$, we have a transformation $\phi : v \to v'$ such that $\mathbb{F}(\phi) = \mathbb{F}(v); \mathfrak{r}_A$. In this case, we say that ϕ is an *identification.* We say that $\mathbb{F}$ *strictly reflects identities* when all the corresponding identifications can be chosen to be identity transformations.

Proposition 5.3.0.6. *A homomorphism of globular multicategories with pre-homomorphism types $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ is an acyclic fibration if and only if all the following conditions hold:*

(i) The homomorphism $\mathbb{F}$ has the right lifting property against the boundary-inclusions of types.

(ii) The homomorphism $\mathbb{F}$ is surjective on terms.

(iii) The homomorphism $\mathbb{F}$ weakly reflects identities of terms.

Proof. First suppose that $\mathbb{F}$ is an acyclic fibration. Then (i) follows trivially. For each generic type (or term) $\mathbb{U}$, the unique map $\emptyset \to \mathbb{U}$ is a cofibration. The lifting property of $\mathbb{F}$ with respect to this map tells us that $\mathbb{F}$ is surjective on types or terms with the same shape as $\mathbb{U}$. This proves (ii).

Now suppose that $v, v' : \Gamma \to A$ are π-shaped parallel n-terms in $\mathbb{X}$ and that $\mathbb{F}(v) = \mathbb{F}(v')$. Then, v and v' together correspond to a homomorphism $[v, v'] : \partial\mathbb{I}_\pi^{\mathcal{H}_n} \to \mathbb{X}$. Furthermore, we have the following commutative square:

$$\begin{tikzcd}
\partial\mathbb{I}_\pi^{\mathcal{H}_n} \arrow[r, "{[v,v']}"] \arrow[d, "\partial\mathbb{I}_\pi^{\mathcal{H}_n}"'] & \mathbb{X} \arrow[d, "\mathbb{F}"] \\
\mathbb{I}_\pi^{\mathcal{H}_n} \arrow[r, "\mathbb{F}(v);\mathfrak{r}_A"'] \arrow[ur, dotted] & \mathbb{Y}
\end{tikzcd}$$

Since $\mathbb{F}$ is an acyclic fibration, this square has a filler. This filler defines the transformation $v \to v'$ required by (iii).

Now suppose on the other hand that we have (i), (ii), and (iii). Let $\mathbb{I}_\pi^n$ be the generic π-shaped n-term, and fix a commutative square:

$$\begin{tikzcd}
\partial\mathbb{I}_\pi^n \arrow[r, "\widetilde{\partial v}"] \arrow[d, "\iota_{\mathbb{I}_\pi^n}"'] & \mathbb{X} \arrow[d, "\mathbb{F}"] \\
\mathbb{I}_\pi^n \arrow[r, "v"'] & \mathbb{Y}
\end{tikzcd}$$

Suppose that $\widetilde{\partial v} : \widetilde{sv} \;\nrightarrow \widetilde{tv}$. By (ii), there is a π-shaped n-term w in $\mathbb{X}$ such that $\mathbb{F}(w) = v$. It follows that $\mathbb{F}(sw) = sv = \mathbb{F}(\widetilde{sv})$ and $\mathbb{F}(tw) = tv = \mathbb{F}(\widetilde{tv})$. Hence, by (iii) there are transformations $\phi : \widetilde{sv} \to sw$ and $\psi : tw \to \widetilde{tv}$ such that $\mathbb{F}(\phi) = sw; r$ and $\mathbb{F}(\psi) = tw; \mathfrak{r}$. We define

$$\tilde{v} = \phi \circ w \circ \psi$$

By construction $\partial\tilde{v} = \widetilde{\partial v}$. Furthermore, homomorphisms of globular multicategories with pre-homomorphism types preserve $- \circ -$, and so $\mathbb{F}(\tilde{v}) = \mathbb{F}(w) = v$. Hence, $\tilde{v}$ defines the required filler, and so $\mathbb{F}$ is an acyclic fibration. □

5.4 Weak Higher Categories from Fibrational Homomorphism Types

Acyclic fibrations allows us to weaken the theories described by globular multicategory. Suppose that $\mathbb{X}$ is a globular multicategory, and that that $\mathbb{Y} \twoheadrightarrow \mathbb{X}$ is an acyclic fibration. Then we can think of $\mathbb{Y}$ as a *weakening of* $\mathbb{X}$. Suppose, for example, that

$\mathbb{X}$ and $\mathbb{Y}$ are operads, and that $\mathbb{X} = \mathbb{1}$ is the terminal globular operad, the theory of strict ω-categories. By Example 5.3.0.4, $\mathbb{Y}$ is contractible, and so, in a sense, parameterises a theory of weak ω-categories.

Consider the following diagram of adjunctions whose right adjoints are forgetful functors:

$$\begin{tikzcd}
\mathrm{GlobMult} \arrow[r, bend left, "L^{\mathrm{Fib}}_{\mathcal{H}}"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult}^{\mathrm{Fib}}_{\mathcal{H}} \arrow[l, bend left, "U^{\mathrm{Fib}}_{\mathcal{H}}"] \arrow[r, bend left, "S"] \arrow[r, phantom, "\perp"] & \mathrm{GlobMult}_{\mathcal{H}} \arrow[l, bend left, "U"]
\end{tikzcd}$$

We refer to the left-hand adjunction as the *strictification* adjunction. Its left adjoint S adds identities of the form $\mathfrak{J}_x(\mathfrak{r}_x; f) = f$ to globular multicategories with fibrational homomorphism types. Let $\eta : \mathrm{id} \Rightarrow US$ be the unit of this adjunction. The following result allows us to view structures in globular multicategories with fibrational homomorphism types as weakenings of structures in globular multicategories with strict homomorphism types.

Theorem 5.4.0.1. *Given any globular multicategory $\mathbb{X}$, the strictification unit*

$$\begin{tikzcd}
L^{\mathrm{Fib}}_{\mathcal{H}}\mathbb{X} \arrow[d, two heads, "\eta(L^{\mathrm{Fib}}_{\mathcal{H}}\mathbb{X})"] \\
USL^{\mathrm{Fib}}_{\mathcal{H}}\mathbb{X} \arrow[d, equal] \\
UL_{\mathcal{H}}\mathbb{X}
\end{tikzcd}$$

is an acyclic fibration.

In order to prove this theorem, we first need to introduce a new notion.

Definition 5.4.0.2. We say that a context is *reduced* when there does not exist a homomorphism type $\mathcal{H}_M$ in Γ. Each context Γ induces a reduced context Γ_ν such that

$$\Gamma = \Gamma_\nu \oplus_{x_1} \mathcal{H}_{B_1} \oplus_{x_2} \cdots \oplus_{x_l} \mathcal{H}_{B_l}$$

for some sequence of variables $x_1 : B_1, \ldots, x_l : B_l$. We say that a term is reduced when its source context is reduced. Composing with reflexivity terms, it follows that every term $f : \Gamma \to A$ induces a canonical reduced term f_ν such that $\mathfrak{r}_{S_\nu}; f = f_\nu$. The laws defining homomorphism types in a globular multicategory with fibrational homomorphism types ensure that f_ν is well-defined.

Proposition 5.4.0.3. *Suppose that $F : \mathbb{X} \to \mathbb{Y}$ is a homomorphism of globular multicategories with homomorphism types, and that the homomorphism types of $\mathbb{Y}$ are strict. Suppose that $f, f' : \Gamma \to A$ are parallel terms in $\mathbb{X}$ such that $F(f) = F(f')$. Suppose that we have a transformation $\phi_\nu : f_\nu \to f'_\nu$ such that $F(\phi_\nu) = F(f_\nu);\mathfrak{r}_A$. Then there exists a transformation $\phi : f \to f'$ in $\mathbb{X}$ such that $F(\phi) = F(f);\mathfrak{r}_A$ and $\mathfrak{r}_{S_\nu};\phi = \phi_\nu$. Furthermore, when $\mathbb{X}$ has strict homomorphisms, we have that $\phi = f;\mathfrak{r}_A$.*

Proof. We have that $\Gamma = \Gamma_\nu \oplus_{x_1} \mathcal{H}_{B_1} \oplus_{x_2} \cdots \oplus_{x_l} \mathcal{H}_{B_l}$ for some sequence of variables $x_1 : B_1, \ldots, x_l : B_l$. Hence, since F has strict homomorphism types, we have that

$$\begin{aligned} F(f);\mathfrak{r}_A &= \mathfrak{J}_{x_l} \cdots \mathfrak{J}_{x_1}(F(f)_\nu);\mathfrak{r}_A \\ &= \mathfrak{J}_{x_l} \cdots \mathfrak{J}_{x_1}(F(f_\nu));\mathfrak{r}_A \\ &= \mathfrak{J}_{x_l} \cdots \mathfrak{J}_{x_1}(F(f_\nu);\mathfrak{r}_A) \\ &= \mathfrak{J}_{x_l} \cdots \mathfrak{J}_{x_1}(F(\phi_\nu)) \end{aligned}$$

Hence, repeatedly applying J-rules, we obtain a term

$$\phi = \mathfrak{J}_{x_l} \cdots \mathfrak{J}_{x_1}(\phi_\nu)$$

in $\mathbb{X}$ such that $\phi = F(f);\mathfrak{r}_A$ as required. The "furthermore" part follows immediately. □

Proof of Theorem 5.4.0.1. It is easily seen that $\eta(L_{\mathcal{H}}\mathbb{X})$ is surjective on types and terms. Hence, it suffices to show that it weakly reflects identifications. A straightforward induction shows that each reduced term in $L_{\mathcal{H}}\mathbb{X}$ is of the form $\eta_{\mathcal{H}}(\mathbb{X})(g);r$, for a unique term g in $\mathbb{X}$ and a unique composite reflexivity term $r : A \to \mathcal{H}^k A$. Similarly, each reduced term in $L_{\mathcal{H}}\mathbb{X}$ is of the form $\eta_{\mathcal{H}}(\mathbb{X})(g);r$ for a unique choice of a g and r. Furthermore, the homomorphism $\eta(L_{\mathcal{H}}\mathbb{X})$ sends $\eta_{\mathcal{H}}(\mathbb{X})(g);r$ to $\eta_{\mathcal{H}}(\mathbb{X})(g);r$. Hence, since $\eta_{\mathcal{H}}(\mathbb{X})$ is injective on terms, the homomorphism $\eta(L_{\mathcal{H}}\mathbb{X})$ is injective on reduced terms. The result now follows from Proposition 5.4.0.3. □

Remark 5.4.0.4. The constructed acyclic fibration is easily seen to be bijective on 0-terms. This is an analogue of the normalisation condition which is frequently placed on globular operads.

Example 5.4.0.5. By Example 5.2.0.2, we have that $L_{\mathcal{H}}\mathbb{T}_0 = \mathbb{1}$, the terminal globular operad. Thus, in this case, Theorem 5.4.0.1 tells us that $L^{\mathrm{Fib}}_{\mathcal{H}}\mathbf{T}_0$ is a contractible globular operad. Remark 5.4.0.4 tells us that this operad is normalised. It follows that every 0-type in a globular multicategory with fibrational homomorphism has the structure of a weak ω-category.

Example 5.4.0.6. By, Example 5.2.0.4, the theory of strict ω-functors is $L_{\mathcal{H}}\mathbb{I}_0^0$. Hence, $L_{\mathcal{H}}^{\mathrm{Fib}}$ can be seen as a theory of weak ω-functors. Every 0-term in a globular multicategory with fibrational homomorphism types is a weak ω-functor in this sense.

Example 5.4.0.7. By Example 5.2.0.3, the theory of strict ω-profunctors is $L_{\mathcal{H}}\mathbb{T}_1$. Hence, $L_{\mathcal{H}}^{\mathrm{Fib}}\mathbb{T}_1$ can be seen as a theory of weak ω-profunctors. Every 1-type in a globular multicategory with fibrational homomorphism types is a weak ω-profunctor in this sense.

Example 5.4.0.8. Analogous to the strict case, the higher categorical structures on types endow globular sets of terms

$$[[\Gamma \longrightarrow M, \quad g \relbar\joinrel\mapstochar\joinrel\rightarrow h]]$$

with the structure of a weak higher category.

Since these results all hinge on Theorem 5.4.0.1, the following conjecture would allow us to prove similar results about globular multicategory with weak homomorphism types:

Conjecture 5.4.0.9. *Given any globular multicategory $\mathbb{X}$, the strictification unit*

$$\begin{tikzcd}
L_{\mathcal{H}}^{\mathrm{Wk}}\mathbb{X} \arrow[d, two heads, "\eta(L_{\mathcal{H}}^{\mathrm{Wk}}\mathbb{X})"] \\
USL_{\mathcal{H}}^{\mathrm{Wk}}\mathbb{X} \arrow[d, equal] \\
UL_{\mathcal{H}}\mathbb{X}
\end{tikzcd}$$

is an acyclic fibration.

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